DIV, GRAD, CURL AND ALL THAT………

Scalar Functions e.g. Temperature T (x, y, z)

Function of (x, y, z)

Vector Functions:

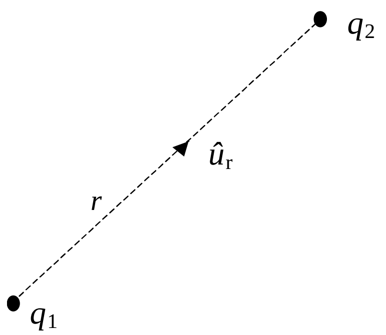
$$\mathbf{F}(x, y, z) = \mathbf{i}\, F_x(x, y, z) + \mathbf{j}\, F_y(x, y, z) + \mathbf{k}\, F_z(x, y, z)$$

F_x F_y and F_z are scalar functions

Specifies magnitude and direction e.g. Velocity of a fluid

Coulombs Law
(in vacuum)

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{u}_r$$



- Products of charges
- $1/r^2$

:- Acts along line joining q_1 and q_2

q_1 and q_2

Principles of superposition

If $\mathbf{F}_1$ is the force exerted on q_0 by q_1 when there are no other charges nearby, and $\mathbf{F}_2$ is the force exerted on q_0 by q_2 when there are no other charges nearby, then the principle of superposition states that the net force exerted on q_0 by q1 and q_2 when they are both present is the vector sum $\mathbf{F}_1 + \mathbf{F}_2$.

- All forces are vectorally.
- Force between two charged particles is not modified by the presence of other charges.

Electric field at **r** due to the charge q_1

$$\mathbf{E}(r) = \frac{\mathbf{F}(\mathbf{r})}{q_2} = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r^2} \hat{\mathbf{u}} r$$

For a group of charges

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=1}^{N}\frac{q_0 q_1}{|\mathbf{r}-\mathbf{r}_l|^2}\,\hat{\mathbf{u}}\mathbf{r}_l r$$

Force on q_0 at $\mathbf{r} = i\mathbf{x} + \mathbf{j}y + k\mathbf{z}$

due to charges q_l at r_l

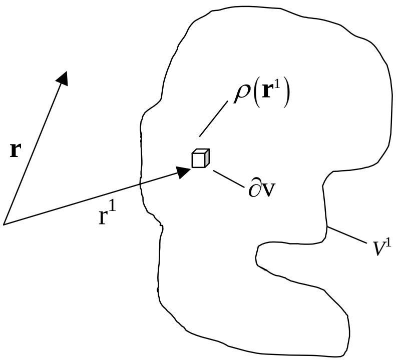
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=1}^{N}\frac{q_l}{|\mathbf{r}-\mathbf{r}_l|^2}\,\hat{u}\mathbf{r}_1, r$$

Electrostatic field at $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

due to charges q_l at r_l

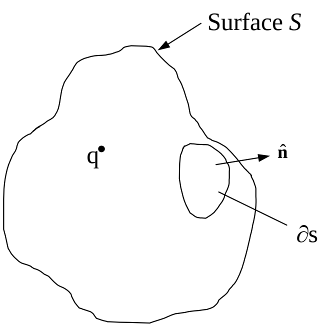
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{V^1}\frac{\rho(\mathbf{r}^1)\hat{\mathbf{u}}r^1, r\,dv^1}{|\mathbf{r}-\mathbf{r}^1|^2}$$

For continuous distribution of charge.



$$Q = \iiint_{V^1}\rho(\mathbf{r}^1)\partial v^1$$

SURFACE INTEGRALS AND THE DIVERGENCE.

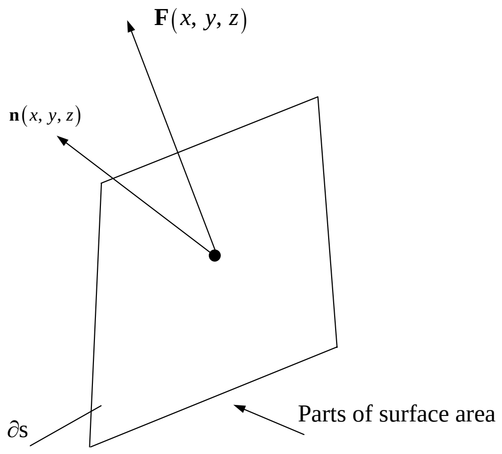


Gauss' Law

$$\oint_S \mathbf{E}\bullet\hat{\mathbf{n}}\,\partial s = \frac{q}{\varepsilon_0}$$

$\hat{\mathbf{n}}$: Unit vector normal to the surface.

SURFACE INTEGRAL

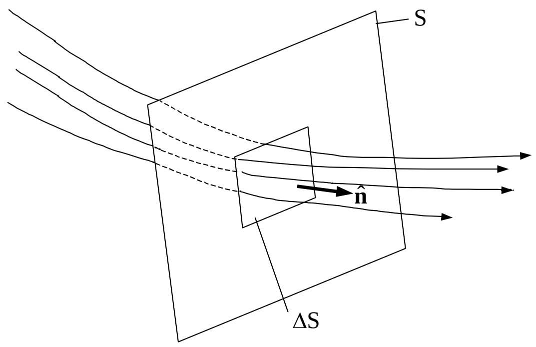


$$\int_S \mathbf{F}(x, y, z) \bullet \mathbf{n}(x, y, z) \partial s = \text{"Flux of F"}$$

Can be open or closed surface

For closed surface points outward from the volume enclosed by the surface

Mass flow



Rate of flow = $\rho \mathbf{v} \bullet \mathbf{n} \Delta S$
through ΔS

Total flux = $\int_S \rho \mathbf{v} \bullet \mathbf{n} \partial s$

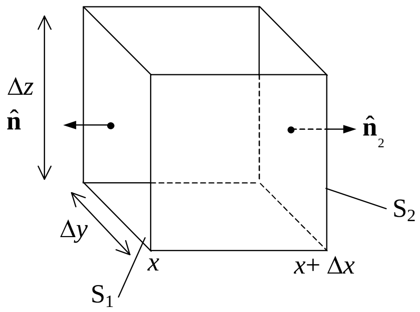
ρ : Density V : Velocity of mass flow.

THE DIVERGENCE

$$div\, \mathbf{F} = \nabla \bullet \mathbf{F} = \begin{matrix} Lim \\ \Delta V \to 0 \\ \text{about} \\ (x, y, z) \end{matrix} \frac{1}{\Delta V} \oint_S \mathbf{F} \bullet \hat{\mathbf{n}} \partial s$$

Scalar quantity which is a function of position (x, y, z)

Consider a little box



$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_2 \mathbf{k}$$

$$F_{x+\Delta x} = F_x + \frac{\partial F_x}{\partial x}\Delta x$$

$$\int_{S_1+S_2} \mathbf{F} \bullet \hat{\mathbf{n}}\, \partial s \;\; \mathrm{Y} \;\; \left[Fx + \frac{\partial Fx}{\partial x}\Delta x - Fx \right] \Delta z \Delta y$$

$$= \frac{\partial F_x}{\partial x}\Delta x \Delta z \Delta y = \frac{F_x}{\partial x}\Delta V$$

Do the same for the other faces

$$\Rightarrow \frac{1}{\Delta V}\oint_S \mathbf{F} \bullet \hat{\mathbf{n}}\, \partial s = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \bullet \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

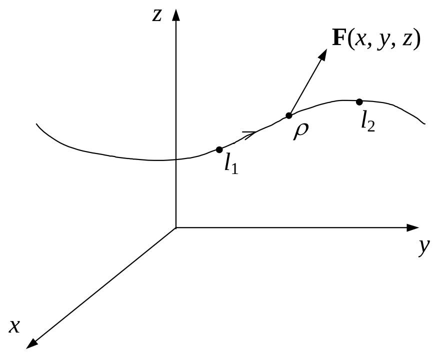
DIVERGENCE THEOREM

$$\boxed{\oint_S \mathbf{F} \bullet \hat{\mathbf{n}}\, \partial s = \int_V \nabla \bullet \mathbf{F}\, \partial V}$$

The flux of a vector function through a closed surface equals the (triple) integral of the divergence of that function over the volume enclosed by the surface.

F must be continuous, differentiable and its first derivatives are continuous in *V* and on *S*.

LINE INTEGRALS INVOLVING VECTOR FUNCTIONS



Work done by force in moving a particle along the curve from l_1 to l_2

$$\omega = \int_{\substack{c \\ l_1 \to l_2}} \mathbf{F}\left(x, y, z\right) \bullet \hat{\mathbf{t}}\,\partial l$$

$$\hat{\mathbf{t}}\left(l\right) = \mathbf{i}\frac{\partial x}{\partial l} + \mathbf{j}\frac{\partial y}{\partial l} + \mathbf{k}\frac{\partial z}{\partial l}$$

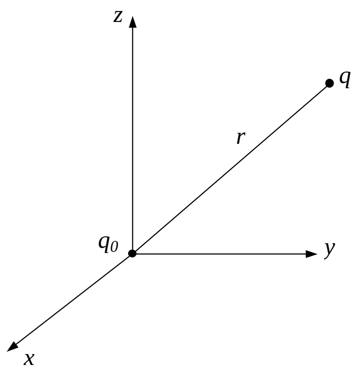
$\hat{\mathbf{t}}$ unit vector tangential to curve at point P

(only the component that acts along path does work)

$$\therefore \quad \omega = \int_C F_x \partial x + F_y \partial y + F_z \partial z$$

The value of a line integral can (and usually does) depend on the path of integration.

Path independence of work done by Coulomb Force.



Coulomb Force on q

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{\mathbf{u}}$$

$$\hat{\mathbf{u}} = \frac{r}{|\mathbf{r}|} = \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{r}$$

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} qq_0\left(\frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}z}{r^3}\right) \qquad \hat{\mathbf{t}}\partial l = \mathbf{i}\partial x + \mathbf{j}\partial y + \mathbf{k}\partial z$$

$$\int_C \mathbf{F} \bullet \hat{\mathbf{t}}\partial l = \frac{qq_0}{4\pi\varepsilon_0}\int_{r_1}^{r_2} \frac{x\partial x + y\partial y + z\partial z}{r^3}$$

$$\begin{bmatrix} r^2 = x^2 + + y^2 + z^2 \\ r\partial r = x\partial x + y\partial y + z\partial z \end{bmatrix} = \frac{qq_0}{4\pi\varepsilon_0}\int_{r^1}^{r^2} \frac{\partial r}{r^2}$$

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$$\omega = \frac{qq_0}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Haven't had to specify C! Get same answer whatever path.

Have considered only one charge. If there are many $q_0, q_1, q_2 \ldots . q_N$ then total force on q

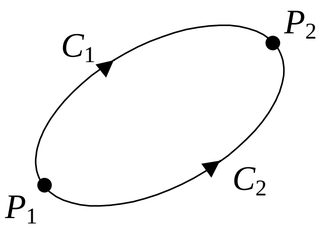
$$\mathbf{F}_T = \mathbf{F}_0 + \mathbf{F}_1 + \mathbf{F}_2 + \ldots .. + \mathbf{F}_N$$

$$\therefore \int_C \mathbf{F} \bullet \hat{\mathbf{t}}\partial l = \int_C \mathbf{F}_0 \bullet \hat{\mathbf{t}}\partial l + \int_C \mathbf{F}_1 \bullet \hat{\mathbf{t}}\partial l + \ldots .. + \int \mathbf{F}_N \bullet \hat{\mathbf{t}}\partial l$$

(Principle of superposition)

Coulomb force depends only on distance between two particles and acts along the line joining them $\leftarrow$ Central force.

For any central force $\int_C \mathbf{F} \bullet \hat{\mathbf{t}}\partial l$ is path independent



$$\int_{C_1} \mathbf{F} \bullet \hat{\mathbf{t}}\partial l = \int_{C_2} \mathbf{F} \bullet \hat{\mathbf{t}}\partial l = -\int_{C_2} \mathbf{F} \bullet \hat{\mathbf{t}}\partial l$$

$$\Rightarrow \oint \mathbf{F} \bullet \hat{\mathbf{t}}\partial l = 0$$

Since $\mathbf{F} = q\mathbf{E} \quad \Rightarrow \int_C \mathbf{E} \bullet \hat{\mathbf{t}}\partial l$ is path independent

And $\oint_C \mathbf{E} \bullet \hat{\mathbf{t}}\partial l = 0$

Conservative field

Curl F = ∇ × F

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Stokes Theorem

(**F** could be any vector field)

$$\boxed{\oint_C \mathbf{F} \bullet \hat{\mathbf{t}} \partial l = \int_S \hat{\mathbf{n}} \bullet \nabla \times \mathbf{F}\, \partial s}$$

The line integral of the tangential component of a vector function over some closed path equals the surface integral of the normal component of the curl of that function integrated over any capping surface of the path.
F Must be continuous and be differentiable and have continuous derivatives on C and S.

The meaning of curl

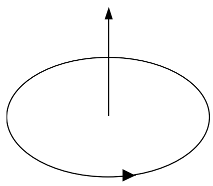
Think of water draining from a bathtub → not quite going to do this but → consider a small volume of water at (x, y) where

$$x = r\cos\omega t$$
$$y = r\sin\omega t$$

$$\mathbf{V}(x, y) = \mathbf{i}\frac{\partial x}{\partial t} + \mathbf{j}\frac{\partial y}{\partial t} = \omega\left[-y\mathbf{i} + x\mathbf{i}\right]$$

↑

velocity field $\nabla \times \mathbf{V} = 2\omega\mathbf{k}$



Consider Electrostatic field

$$\oint_C \mathbf{E} \bullet \hat{\mathbf{t}}\, \partial l = 0 = \int_S \hat{\mathbf{n}} \bullet \nabla \times \mathbf{E}\, \partial s$$

$$\boxed{\Rightarrow \nabla \times \mathbf{E} = 0 \qquad \text{for an electrostatic field}}$$

Note: if $\nabla \times \mathbf{E} \neq 0$ E cannot be an electrostatic field

The gradient

Suppose $\mathbf{F}(x, y, z) = F_x\mathbf{i} + F_x\mathbf{j} + F_z\mathbf{k}$

And $F_x = \dfrac{\partial\phi}{\partial x}$ $F_y = \dfrac{\partial\phi}{\partial y}$ $F_2 = \dfrac{\partial\phi}{\partial z}$

Where $\phi(x, y, z)$ is a scalar

$$\mathbf{F} \bullet \hat{\mathbf{t}} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial l} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial l} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial l} = \frac{\partial\phi}{\partial l}$$

$$\hat{\mathbf{t}} = \mathbf{i}\frac{\partial x}{\partial l} + \mathbf{j}\frac{\partial y}{\partial l} + \mathbf{k}\frac{\partial z}{\partial l}$$

$$\therefore \int_C \mathbf{F} \bullet \hat{\mathbf{t}}\,\partial l = \int_C \frac{\partial\phi}{\partial l}\partial l = \phi(x, y, z) - \phi(x_0, y_0, z_0)$$

If **F** and ϕ are related as above then the line integral is independent of path.

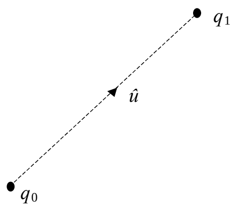
Depends on position of start and finish not path

$$\mathbf{F} = \nabla\phi = \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right]\phi$$

ELECTROSTATICS IN A VACUUM

COULOMBS LAW

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\ \frac{q_0 q_1}{r^2}\ \hat{\mathbf{u}}_r \qquad \begin{bmatrix}\text{FORCE AT } \mathrm{q}_1 \\ \text{DUE TO } \mathrm{q}_0\end{bmatrix}$$



ELECTRIC FIELD STRENGTH

$$\mathbf{E} = \frac{\mathbf{F}}{q_1} \qquad \left[\mathrm{Vm}^{-1}\right] \quad \begin{bmatrix}\text{FIELD AT } q_1 \\ \text{DUE TO } q_0\end{bmatrix}$$

E lines start on +ve charge

end on –ve charge

Electrostatic field **E** is conservative

$\oint_l \mathbf{E} \bullet d\mathbf{l} = 0$ No work done around a closed path $\left[d\mathbf{l} = \hat{\mathbf{t}}dl\right]$

Stokes theorem $\oint_l \mathbf{E} \bullet d\mathbf{l} = \int_S \nabla \times \mathbf{E} \bullet d\mathbf{S}$ $\left[d\mathbf{S} = \hat{\mathbf{n}}\ dS\right]$

Surface bounded by path

$\Rightarrow \nabla \times \mathbf{E} = 0$ $\Rightarrow \mathbf{E} = -\nabla V$ Ensures that lines of **E** start on +ve charge

V = Electric scalar potential [V m^{-1}]

$$\nabla V = \mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}$$

ABSOLUTE POTENTIAL

$V_A = -\int_{\infty}^{A} \mathbf{E} \bullet d\mathbf{l}$ External work done per unit (+ve) charge to move unit test charge from infinity to the point A.

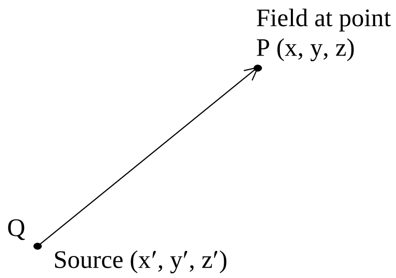
POTENTIAL DIFFERENCE

$V_{BA} = V_B - V_A = -\int_{A}^{B} \mathbf{E} \bullet d\mathbf{l}$ Work done per unit chare in moving unit test charge from $A \rightarrow B$.

CHARGE DISTRIBUTION

a) Point charge Q

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$



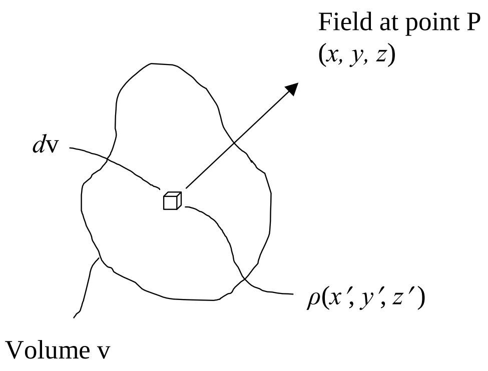
$$V = -\int_{\infty}^{r} \mathbf{E} \bullet d\mathbf{r} = \frac{Q}{4\pi\varepsilon_0 r}$$

$$\mathbf{r} = \mathbf{i}(x - x') + \mathbf{j}(y - y') + \mathbf{k}(z - z')$$

$$r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$$

b) Volume charge distribution $\left[\rho \text{ Cm}^{-3}\right]$



$$\text{E} = \frac{1}{4\pi\varepsilon_0}\int_{\text{v}} \frac{\hat{\mathbf{r}}\rho d\text{v}}{\text{r}^2}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\text{v}} \frac{\rho d\text{v}}{r}$$

$$Q = \int_{\text{v}} \rho d\text{v}$$

OR

$$dQ = \rho d\text{v}$$

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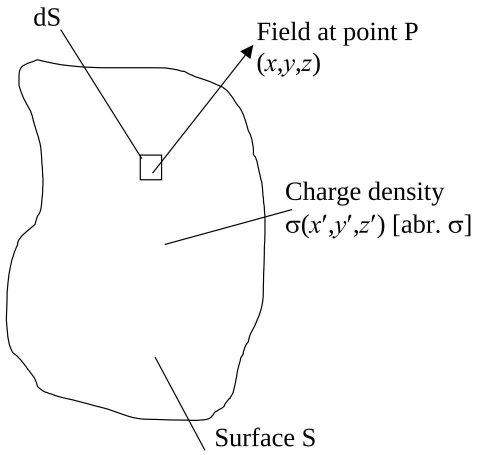
c) Surface charge distribution $\left[\sigma\ \text{Cm}^{-2}\right]$



$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\hat{\mathbf{r}}\sigma dS}{r^2} \qquad Q = \int_S \sigma dS$$

OR

$$V = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma dS}{r} \qquad dQ = \sigma dS$$

Gauss's law: Laplace's and Poisson's equations in vacuum

Have seen $\mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\hat{\mathbf{r}}}{r^2}$ [in vacuum]

$$\oint_S \mathbf{E}\bullet d\mathbf{S} = \int \frac{Q}{4\pi\varepsilon_0}\ \frac{\hat{\mathbf{r}}}{r^2}\bullet d\mathbf{S}$$

$\left[\begin{array}{l}\text{Choose a simple}\\ \text{surface}\end{array}\right]$

$$d\mathbf{S} = \hat{\mathbf{r}}\, r^2 \sin\theta d\theta d\phi$$

$$\oint_S \mathbf{E}\bullet d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{Q}{4\pi\varepsilon_0}\,\frac{r^2\sin\theta d\theta d\phi}{r^2}\left(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}\right)$$

$$\oint_S \mathbf{E}\bullet d\mathbf{S} = 2\pi\times 2\times\frac{Q}{4\pi\varepsilon_0}$$

$$\oint_S \mathbf{E}\bullet d\mathbf{S} = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_{\mathrm{v}} \rho d\mathrm{v}$$

Gauss' Theorem $\oint_S \mathbf{E} \bullet d\mathbf{S} = \int_{\mathrm{v}} \nabla \bullet \mathbf{E}\, d\mathrm{v} = \frac{1}{\varepsilon_0} \int_{\mathrm{v}} \rho d\mathrm{v}$

$\therefore \quad \nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_0}$ Differential form of Gauss' Law in vacuum

$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_0}$ but $\mathbf{E} = -\nabla V$ $\Rightarrow \nabla \bullet \mathbf{E} = \nabla \bullet \left(-\nabla V\right) = -\nabla^2 V$

$$\Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ Poisson's equation

If $\rho = 0$ $\nabla^2 V = 0$ Laplace's equation

One solution is $V = \frac{1}{4\pi\varepsilon_0} \int_{\mathrm{v}} \frac{\rho d\mathrm{v}}{r}$

But to this we must add all possible solutions of homogeneous equations **that** are consistent with the boundary conditions (symmetry) of the problem.

EXAMPLE EXAM STYLE QUESTION

A perfectly conducting sphere is placed in a previously uniform electric field pointing in the *z*- direction. The sphere is uncharged and has a radius *a*.

i) How is the electric field changed?

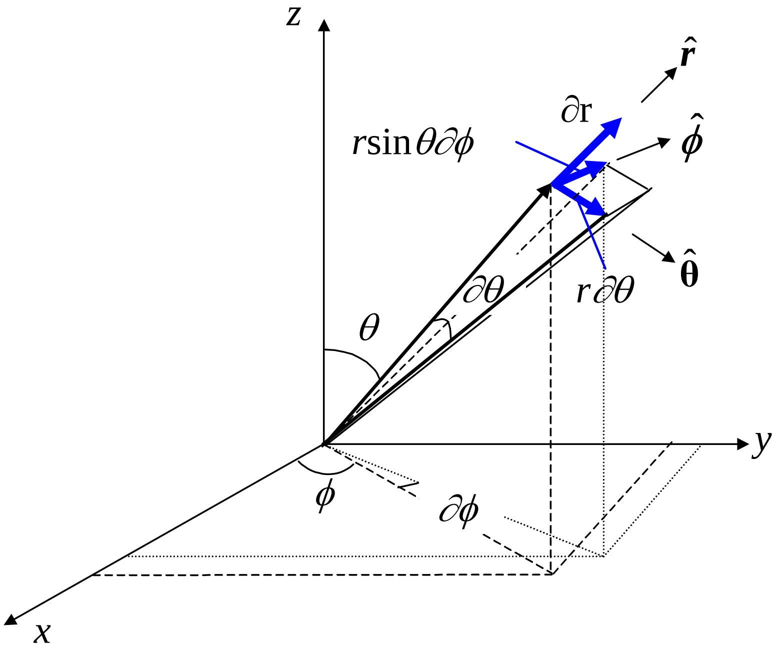
ii) What is the surface density on the sphere?

iii) What is the induced dipole moment of the sphere? (Would not get this last bit in exam!)

THINK ABOUT THE SYMMETRY OF THE PROBLEM!

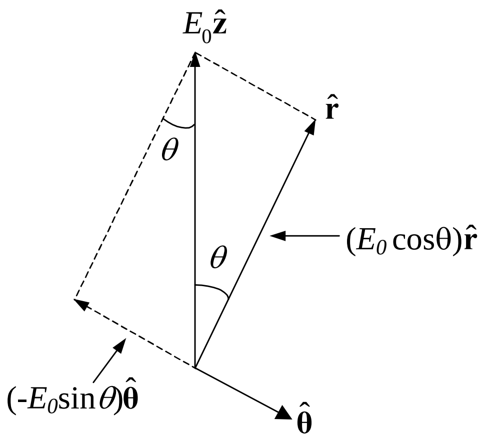
Sphere $\Rightarrow$ We should work in spherical polar coordinates.

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$



We are told that $\mathbf{E} = E_0\,\hat{\mathbf{z}}$ in Cartesian coordinates

How to write this in Spherical Polar Coordinates?



$$\mathbf{E}(r,\theta,\phi) = (E_0\cos\theta)\hat{\mathbf{r}} - (E_0\sin\theta)\hat{\boldsymbol{\theta}}$$

Now since $\mathbf{E} = -\nabla V$

$$\Rightarrow V(r,\theta,\phi) = -E_0 r\cos\theta$$

Now we put sphere into field $\Rightarrow$ field is perturbed

Potential must be a solution of $\nabla^2 V = 0$

Try $V = -E_0 r\cos\theta \quad + \quad \dfrac{A\cos\theta}{r^2}$ with origin at centre of sphere.

N.B. All terms must have same θ dependence to match at boundary.

When $r \to \infty$ must get back to uniform E i.e. $1/r^2 \to 0$

In spherical polar coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

Check and find $V(r,\theta,\phi) = -E_0 r\cos + \dfrac{A\cos\theta}{r^2}$ is a solution of $\nabla^2 V = 0$

N.B. : Charge on a conductor resides on its surface

: Every point on or in a perfect conductor has the same potential.

: If V = constant $\quad E = -\nabla V = 0$ i.e. within conductor E = 0

Initially sphere uncharged $\Rightarrow$ V = 0, no net charge on conductor when moved into field $\Rightarrow$

For $\quad r \le a \qquad V(r,\theta,\phi) = 0$ Boundary condition 1

For $\quad r \to \infty \qquad V(r,\theta,\phi) \to -E_0 r\cos\theta$ 2

Use BC (1) at $r = a \qquad V = 0 \qquad \Rightarrow \quad A = E_0 a^3$

BC (2) $r \to \infty \qquad V \to -E_0 r\cos\theta$

$\therefore$ For $r > a \qquad V(r,\theta,\phi) = -E_0 r\cos\theta + \dfrac{E_0 a^3}{r^2}\cos\theta$ 3

$$\mathbf{E} = -\nabla V = (E_0\cos\theta)\hat{\mathbf{r}} - (E_0\sin\theta)\hat{\boldsymbol{\theta}} + \left(\frac{2E_0 a^3\cos\theta}{r^3}\right)\hat{\mathbf{r}} + \left(\frac{E_0 a^3\sin\theta}{r^3}\right)\hat{\boldsymbol{\theta}}$$

As $r \to a$ $\quad \mathbf{E} = \left(3E_0 \cos\theta\right)\hat{\mathbf{r}} \quad \begin{pmatrix} Lim \\ r \to a \end{pmatrix}$

(From $r > a$)

Gauss' Law

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{Q}{\varepsilon_0} \qquad d\mathbf{S} = \hat{\mathbf{r}} dS$$

$$dS = r^2 \sin\theta d\theta d\phi$$

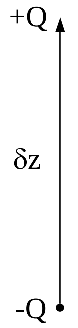
$$\int_S \left(3E_0 \cos\theta\right) dS = \int_S \frac{\sigma dS}{\varepsilon_0}$$

$$\Rightarrow \sigma = 3E_0 \varepsilon_0 \cos\theta$$

Induced Dipole Moment $\quad p = 4\pi\varepsilon_0 a^3 E_0 \quad$ (see next section for why!)

POINT DIPOLE

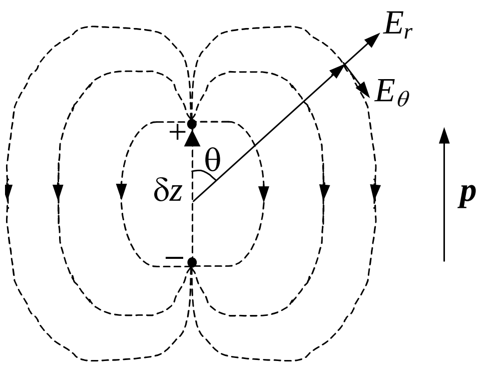
Potential from a dipole: $V_p = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$



Where $|\mathbf{p}| = p = Q\delta z$

$\therefore$ Second term in equation 3 corresponds to the contribution of a dipole with:

$p = 4\pi\varepsilon_0 a^3 E_0$



$\delta z \ll r$

$$V_p = \frac{Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2}$$

$$V_p = \frac{Q}{4\pi\varepsilon_0}\frac{(r_2 - r_1)}{r_1 r_2}$$

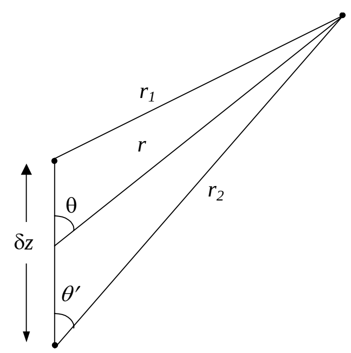
$$V_p = \frac{Q\delta z}{4\pi\varepsilon_0}\frac{(2r_2\cos\theta' - \delta z)}{r_1 r_2 (r_1 + r_2)}$$

Now as $r \gg \delta z$

$\theta' \to \theta \qquad r_1, r_2 \to r$

$$\therefore V_p = \frac{Q}{4\pi\varepsilon_0}\frac{\delta z\cos\theta}{r^2}$$

$$V_p = \frac{1}{4\pi\varepsilon_0}\frac{|\mathbf{p}|\cos\theta}{r^2}$$



$$r_1^2 = r_2^2 + (\delta z) - 2\delta z r_2\cos\theta'$$

$$r_1 - r_2 = \frac{(\delta z)^2 - 2\delta z r_2\cos\theta'}{(r_1 + r_2)}$$

$$V_p = \frac{1}{4\pi\varepsilon_0}\left(\frac{\mathbf{p}\bullet\mathbf{r}}{r^3}\right) = -\frac{1}{4\pi\varepsilon_0}\mathbf{p}\bullet\nabla\left(1/r\right)$$

$\mathbf{E} = -\nabla V$ in spherical polar coordinates $\mathbf{E} = \hat{\mathbf{r}}E_r + \hat{\boldsymbol{\theta}}E_0 + \hat{\boldsymbol{\varphi}}E_\phi$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \qquad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} \qquad E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

THE ELECTROSTATIC PROPERTIES OF DIELECTRIC MATERIALS

IDEAL DIELECTRIC – contains no free charge (perfect insulator)

In practice all material media contain some free charges and therefore have finite conductivity.

DIELECTRIC ⇒ very low electrical conductivity

CLASS 1 DIELECTRIC

Non-Polar media composed of neutral atoms/molecules that have no electric dipole moment in the absence of an applied field.

When an **E** field is applied the electronic orbitals are perturbed. Negative electrons are displaced in a direction opposite to that of the field and positive nuclei tend to move in the same direction as **E**. ⇒ centre of negative charge displaced from centre of positive charge.

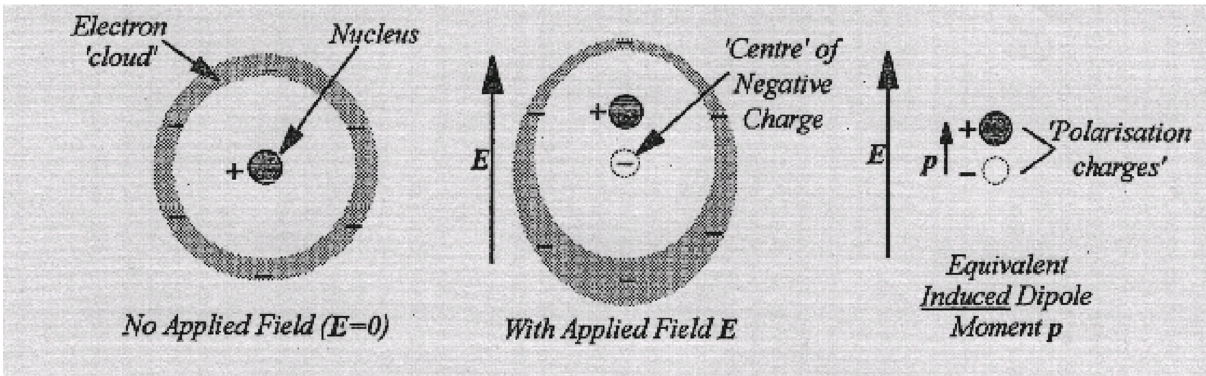


Figure 2.1: Effect of an Electric Field on a Neutral Atom (Class I)

Examples:

	Relative Permittivity	Conductivity
He	$\varepsilon_r = 1.000071$	$<10^{-15}\ \Omega^{-1}m^{-1}$
CH_4	$\varepsilon_r = 1.00098$	$<10^{-15}\ \Omega^{-1}m^{-1}$
Teflon	$\varepsilon_r = 2.0$	$<10^{-15}\ \Omega^{-1}m^{-1}$

CLASS II DIELECTRIC

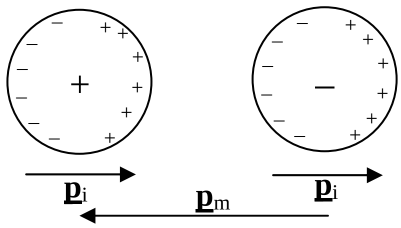
Polar dielectric composed of molecules or ion pair that have a permanent electric dipole moment.

Examples:	H_2O	$\varepsilon_r \approx 80$	@ low frequency and T>273K
	KCl	$\varepsilon_r \approx 5.0$	@ low frequency and T=273K
	NH_3	$\varepsilon_r \approx 1.008$	@ T=273K and 10^5 Pa

Consider a gas of polar molecules with each molecule having a Dipole Moment $\underline{\mathbf{p}}_{\mathbf{m}}$. In the absence of an electric field the directions of these dipoles are randomised by thermal energy. When $\underline{\mathbf{E}}$ applied dipoles tend to align parallel to $\underline{\mathbf{E}}$. The Tendency to align is disturbed by thermal motion. Since $k_BT >> \underline{\mathbf{p}}_{\mathbf{m}}.\underline{\mathbf{E}}$ ($\underline{\mathbf{p}}_{\mathbf{m}}.\underline{\mathbf{E}}$ – electrostatic energy of a dipole). The net moment of a volume of gas is much smaller than it would be if all dipoles were aligned.

When $\underline{\mathbf{E}}$ applied to a Polar Gas also get induced (type 1) dipole moments.

The Electric Field "seen" by each dipole is a combination of the applied field and that due to the other dipoles.



Note get induced dipole even in the absence of applied field. Electrons around one ion see field from charge on other ion.

SOLIDS AND LIQUIDS

TYPE 1 DIELECTRIC – Simple approach works OK

TYPE 2 DIELECTRIC – Complicated

E.G.	H_2O	$\varepsilon_r \approx 80$	for water at $T \geq 273K$
	H_2O	$\varepsilon_r \approx 3$	for ice

In ice permanent dipoles cannot re-orientate!

In ionic solids small displacement of positive and negative ions caused by $\underline{\mathbf{E}}$ gives rise to large electric polarisation and ε_r – see solid-state physics…

DIELECTRIC BREAKDOWN

In E field the few free electric charges in a dielectric are accelerated – if the field large enough then when these electrons collide with atoms (or ions) they produce secondary electrons that are themselves accelerated by $\underline{\mathbf{E}}$.

$\Rightarrow$ AVALANCHE EFFECT – currents flows (in streamers) Dielectric is heated and can be permanently damaged.

Field required for this effect – BREAKDOWN FIELD – typically 10^6 Vm^{-1}

If dielectric thin (say in a commercial capacitor) a few volts can cause breakdown.

DIELECTRIC POLARISATION (***P***) AND ELECTRIC SUSCEPTIBILITY (χ).

For both Class 1 and Class 2 Dielectrics, applied electric field INDUCES an electric dipole moment in each elementary volume of the material. Induced Dipole moment originates from POLARISATION CHARGES – bound to the nuclei and not able to move as free charges.

Macroscopic measure of the induced-dipole effect is the ELECTRICAL POLARISATION ***P***.

P = Induced electric dipole moment / unit volume [Cm^{-2}]

Usually we write $\underline{\boldsymbol{P}} = \varepsilon_0 \chi \underline{\boldsymbol{E}}$

Not always the whole truth! Assumes that

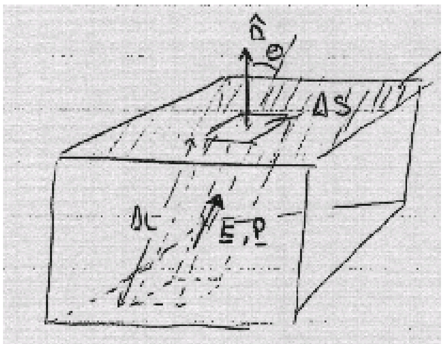
- ***P*** depends linearly on ***E***.
- χ homogeneous
- ***P*** parallel to ***E***.

P IS RELATED TO SURFACE AND BULK POLARISATION CHARGE

When a dielectric acquires an Electric Polarisation ***P*** (by virtue of an internal field ***E***).

(a) A distribution of polarisation charges appears on the surface –surface polarisation charge density $\sigma_p = \underline{\boldsymbol{P}} \bullet \underline{\hat{\boldsymbol{n}}}$ [Cm^{-2}]. $\underline{\hat{\boldsymbol{n}}}$ is outwardly directed unit vector normal to the surface.

(b) A distribution of polarisation charges appears throughout its volume – volume polarisation charge density $\rho_p = -\nabla \bullet \underline{\boldsymbol{P}}$ [Cm^{-3}].



(a) Surface distribution

(i) $\underline{\boldsymbol{E}}$ applied at angle θ to normal to surface $\underline{\hat{\boldsymbol{n}}}$

$\therefore$ $\underline{\boldsymbol{P}}$ at angle θ to $\underline{\hat{\boldsymbol{n}}}$

(ii) Assume polarisation charge $+\Delta Q_P$ on the top ΔS and $-\Delta Q_P$ on the bottom ΔS.

$\therefore$Dipole moment for our volume element = $\Delta l \Delta Q_P = P\Delta \text{v} = \text{P}\Delta l \Delta S \cos\theta$

$$\therefore P\cos\theta = \underline{\boldsymbol{P}} \bullet \underline{\hat{\boldsymbol{n}}} = \frac{\Delta Q}{\Delta S} = \sigma_P$$

$$\Rightarrow \sigma_P = \underline{\boldsymbol{P}} \bullet \underline{\hat{\boldsymbol{n}}}$$

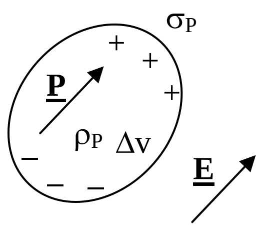
(b) Volume distribution

Consider a small-uncharged volume Δv, and electric field is applied and the material becomes polarised.

Total polarisation charge on surface $= \oint_S \sigma_p dS = \oint_S \underline{\boldsymbol{P}} \bullet \underline{\hat{\boldsymbol{n}}} dS = \oint_S \underline{\boldsymbol{P}} \bullet d\underline{\boldsymbol{S}}$

As Δv was initially uncharged

$$\oint_S \underline{\boldsymbol{P}} \bullet d\underline{\boldsymbol{S}} + \rho_P \Delta\text{v} = 0 \text{ or } \rho_P = -\frac{1}{\Delta\text{v}}\oint_S \underline{\boldsymbol{P}} \bullet d\underline{\boldsymbol{S}}$$



$$\Rightarrow \rho_P = -\nabla \bullet \underline{\boldsymbol{P}}$$

ELECTRIC DISPLACEMENT $\underline{\boldsymbol{D}}$

In free space $\nabla \bullet \varepsilon_0 \underline{\boldsymbol{E}} = \rho_f$ where ρ_f is the free charge density.

If a field E exists in a material medium, the material becomes polarised and polarisation charges are induced with a charge density ρ_P.

Now must modify our equation by including ρ_P so that $\nabla \bullet \varepsilon_0 \underline{\boldsymbol{E}} = \rho_f + \rho_P$ (we note that $\underline{\boldsymbol{E}}$ can begin and end on free and bound charges).

Using $\rho_P = -\nabla \bullet \underline{\boldsymbol{P}}$ $\Rightarrow$ $\nabla \bullet \left(\varepsilon_0 \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}\right) = \rho_f$

We define $\underline{\boldsymbol{D}} = \varepsilon_0 \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$ $\Rightarrow$ $\nabla \bullet \underline{\boldsymbol{D}} = \rho_f$

$\underline{\boldsymbol{D}} =$ ELECTRICAL DISPLACEMENT [Cm^{-2}]

LINES OF $\underline{\boldsymbol{D}}$ CAN ONLY BEGIN AND END ON FREE CHARGES

$\underline{\boldsymbol{D}} = \varepsilon_0 \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$ and $\underline{\boldsymbol{P}} = \varepsilon_0 \chi \underline{\boldsymbol{E}}$ (for linear, homogeneous and isotropic media)

$\Rightarrow \underline{\boldsymbol{D}} = \varepsilon_0 (1 + \chi) \underline{\boldsymbol{E}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}} = \varepsilon \underline{\boldsymbol{E}}$

$\varepsilon_r = (1 + \chi) =$ RELATIVE PERMITTIVITY [Dimensionless]

$\varepsilon =$ ABSOLUTE PERMITTIVITY [Fm^{-1}]

GENERAL FORM OF GAUSS' LAW

$\nabla \bullet \varepsilon_0 \underline{\boldsymbol{E}} = \rho_f + \rho_P$ $\quad \int_{\mathrm{v}} \nabla \bullet \varepsilon_0 \underline{\boldsymbol{E}}\, \mathrm{dv} = \oint_S \varepsilon_0 \underline{\boldsymbol{E}} \bullet \mathrm{d}\underline{\boldsymbol{S}} = \int_{\mathrm{v}} \left(\rho_f + \rho_P\right) \mathrm{dv}$

$\nabla \bullet \underline{\boldsymbol{D}} = \rho_f$ $\quad \int_{\mathrm{v}} \nabla \bullet \underline{\boldsymbol{D}}\, \mathrm{dv} = \oint_S \underline{\boldsymbol{D}} \bullet \mathrm{d}\underline{\boldsymbol{S}} = \int_{\mathrm{v}} \rho_f \, \mathrm{dv}$

GENERAL STATEMENT OF GAUSS' LAW $\oint_S \underline{\boldsymbol{D}} \bullet \mathrm{d}\underline{\boldsymbol{S}} = \int_{\mathrm{v}} \rho_f \, \mathrm{dv}$

WE STILL HAVE $\underline{\boldsymbol{E}} = -\nabla V$ BUT POISSON'S EQUATION BECOMES

$$\nabla^2 V = -\frac{\left(\rho_f + \rho_P\right)}{\varepsilon_0}$$

FIELDS NEAR A CHARGED CONDUCTOR

Lines of $\underline{\boldsymbol{D}}$ and $\underline{\boldsymbol{E}}$ are normal to the surface close to the surface (we have seen this before and will see it again with boundary conditions).

Gauss's Law $\oint_S \underline{\mathbf{D}} \bullet d\underline{\mathbf{S}} = \int_S \sigma_f dS$ since all free charge on surface.

Just above surface $D_n \Delta S = \sigma_f \Delta S$ and $D_n = \sigma_f$ and $E_n = \dfrac{\sigma_f}{\varepsilon_0}$

$E_n = 0$ in conductor since potential everywhere in conductor is the same – uniform potential.

EXAMPLES CONCERNING POLARISATION CHARGES

(a) Relation between ρ_P and ρ_f for a simple linear, homogeneous medium $\Rightarrow$ $\varepsilon_r = \text{constant}$

$\underline{\boldsymbol{D}} = \varepsilon_0 \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$ and $\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}}$

$$\therefore \underline{\boldsymbol{P}} = \underline{\boldsymbol{D}} - \frac{\varepsilon_0}{\varepsilon_0 \varepsilon_r} \underline{\boldsymbol{D}} = \underline{\boldsymbol{D}} \left(\frac{\varepsilon_r - 1}{\varepsilon_r} \right)$$

$\Rightarrow \nabla \bullet \underline{\boldsymbol{P}} = \left(\dfrac{\varepsilon_r - 1}{\varepsilon_r} \right) \nabla \bullet \underline{\boldsymbol{D}}$ and since $\nabla \bullet \underline{\boldsymbol{D}} = \rho_f$ and $\rho_P = -\nabla \bullet \underline{\boldsymbol{P}}$ then

$\rho_P = -\left(\dfrac{\varepsilon_r - 1}{\varepsilon_r} \right) \rho_f$. So if $\rho_f = 0$ then $\rho_P = 0$ $\Rightarrow$ only a surface charge distribution σ_P exists on polarised medium.

EFFECTIVE CHARGE DENSITY

$\nabla \bullet \varepsilon_0 \underline{\boldsymbol{E}} = \rho_f + \rho_P$ so in linear homogeneous medium $\rho_P = -\left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right)\rho_f$ then

$\nabla \bullet \varepsilon_0 \underline{\boldsymbol{E}} = \frac{\rho_f}{\varepsilon_r}$. $\frac{\rho_f}{\varepsilon_r}$ is called the effective charge density.

$\therefore$ If a point charge Q is placed in an dielectric medium the effective charge is $\frac{Q}{\varepsilon_r}$ which is less than Q since $\varepsilon_r > 1$. Physical Reason: On the surface of the dielectric adjacent to the point charge Q there is a surface distribution of polarisation charge of the opposite sign to Q – reducing the effective charge (see the problem sheet).

BOUNDARY CONDITIONS IN ELECTROSTATICS

☺ THE NORMAL COMPONENT OF $\underline{\boldsymbol{D}}$ IS CONTINUOUS ACROSS A BOUNDARY PROVIDED THAT NO FREE CHARGE IS PRESENT ON THE BOUNDARY.

☺ THE TANGENTIAL COMPONENT OF $\underline{\boldsymbol{E}}$ IS CONTINUOUS ACROSS A BOUNDARY.

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ANY SOLUTION TO AN ELECTROSTATICS PROBLEM MUST SATISFY THE BOUNDARY CONDITIONS.
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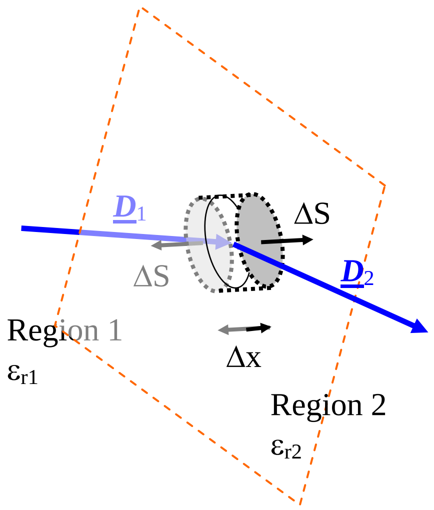
THE NORMAL COMPONENT OF $\underline{\boldsymbol{D}}$



Assume that the free surface charge density is σ_f [Cm^{-2}] on the interface between region 1 (ε_1) and region 2 (ε_2). In region 1 the electric displacement ($\underline{\boldsymbol{D}}_1$) make an angle of θ_1 with the normal to the interface, and in region 2 $\underline{\boldsymbol{D}}_2$ makes an angle of θ_2 with the normal to the interface. Therefore the magnitudes of the components of $\underline{\boldsymbol{D}}$ normal to the interface are $D_{1n} = D_1 \cos\theta_1$ and $D_{2n} = D_2 \cos\theta_2$.

If we apply Gauss's Law to the little "pill box" and let the width of the box $\Delta x \to 0$ then

$$\oint_S \underline{\boldsymbol{D}} \bullet \mathrm{d}\underline{\boldsymbol{S}} = \int_S \sigma_f \mathrm{d}\underline{\boldsymbol{S}}$$

$$D_{2n}\Delta S - D_{1n}\Delta S = \sigma_f \Delta S$$

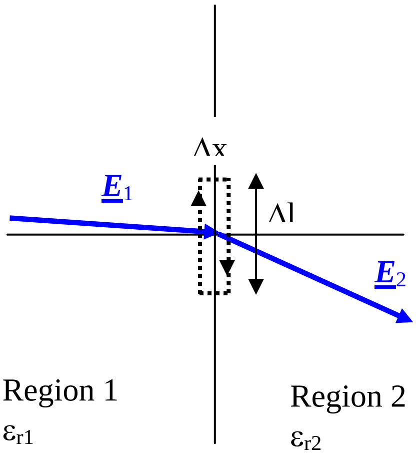
$\therefore D_{2n} - D_{1n} = \sigma_f$ but if $\sigma_f = 0$ then

$$D_{2n} = D_{1n}$$

and because $\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}}$ then there is a "jump" in the normal component of the electric field at the boundary.

$$\varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$$

THE TANGENTIAL COMPONENT OF $\underline{\boldsymbol{E}}$



In region 1 the electric field ($\underline{\boldsymbol{E}}_1$) makes an angle of θ_1 with the normal to the interface, and in region 2 $\underline{\boldsymbol{E}}_2$ makes an angle of θ_2 with the normal to the interface. Therefore the magnitudes of the components of $\underline{\mathbf{E}}$ tangential to the interface are $E_{1t} = E_1 \sin\theta_1$ and $E_{2t} = E_2 \sin\theta_2$.

The line integral of $\underline{\boldsymbol{E}}$ around any closed path in an electrostatic field is zero $\oint_l \underline{\boldsymbol{E}} \bullet d\underline{\boldsymbol{l}} = 0$

So as $\Delta x \to 0$ $\oint_l \underline{\mathbf{E}} \bullet d\underline{\mathbf{l}} = E_{2t}\Delta l - E_{1t}\Delta l = 0$

$\therefore$ $$E_{2t} = E_{1t}$$

(The tangential component of $\underline{\boldsymbol{E}}$ is continuous across a boundary)

Because $\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}}$

$$\frac{D_{2t}}{\varepsilon_2} = \frac{D_{1t}}{\varepsilon_1}$$

Note $\underline{\boldsymbol{E}} = 0$ in a perfect conductor $\Rightarrow$ $E_t = 0$ on surface and the only non-zero component is normal to the surface.

REFRACTION OF LINES OF $\underline{\boldsymbol{D}}$ AND $\underline{\boldsymbol{E}}$.

Boundary conditions:

$E_{1t} = E_{2t}$ or $E_1 \sin\theta_1 = E_2 \sin\theta_2$

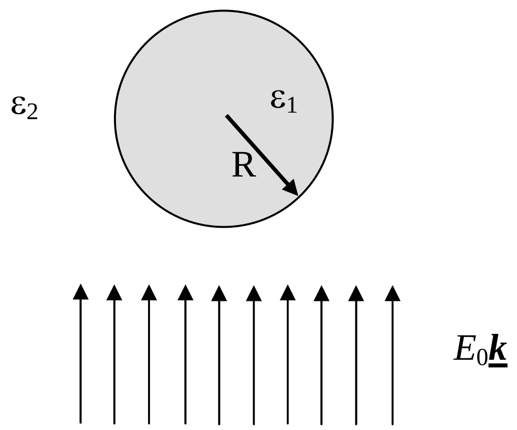
and

$D_{2n} = D_{1n}$ or $D_1 \cos\theta_1 = D_2 \cos\theta_2$

So $\dfrac{D_1}{E_1}\cot\theta_1 = \dfrac{D_2}{E_2}\cot\theta_2$ and since $\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}}$ we find that

$$\varepsilon_1 \cot\theta_1 = \varepsilon_2 \cot\theta_2$$: Refraction formula for field lines.

DIELECTRIC SPHERE IN A UNIFORM FIELD.



Sphere $\Rightarrow$ work in spherical polar coordinates (r,θ,ϕ), but remember that there will be no variation with respect to ϕ.

Boundary Conditions:

Normal component of $\underline{\boldsymbol{D}}$ is continuous: $D_{2r} = D_{1r} \Rightarrow \varepsilon_2 E_{2r} = \varepsilon_1 E_{1r}$
(D_r : is the radial component of $\underline{\boldsymbol{D}}$)

Tangential component of $\underline{\boldsymbol{E}}$ is continuous: $E_{2\theta} = E_{1\theta}$
(E_θ : is the component of $\underline{\boldsymbol{E}}$ tangential to the surface of the sphere)

Try the potentials $V_2 = -E_0 r\cos\theta + \frac{A}{r^2}\cos\theta$ and $V_1 = B_1 r\cos\theta + \frac{B_2}{r^2}\cos\theta$

But as $r \to 0$ in region 1 $V_1 \to \infty$ which means that $B_2 = 0$

So $\quad V_1 = B_1 r\cos\theta$

and $\quad V_2 = -E_0 r\cos\theta + \frac{A}{r^2}\cos\theta$

Also V must be continuous at the boundary (any discontinuity $\Rightarrow$ infinite electric field!)

$\therefore -E_0 R + \frac{A}{R^2} = B_1 R \qquad$ or $\qquad B_1 = -E_0 + \frac{A}{R^3}$

Normal components of $\underline{\boldsymbol{E}}$ at interface are

$$E_{1r} = -\left[\frac{\partial V_1}{\partial r}\right]_{r=R} = -B_1\cos\theta \quad E_{2r} = -\left[\frac{\partial V_2}{\partial r}\right]_{r=R} = E_0\cos\theta + \frac{2A}{R^3}\cos\theta$$

We know $\varepsilon_2 E_{2r} = \varepsilon_1 E_{1r}$ so

$$-\varepsilon_1 B_1 = \varepsilon_2 E_0 + \frac{2\varepsilon_2 A}{R^3} \quad \text{or} \quad B_1 = -\frac{\varepsilon_2}{\varepsilon_1}\left[E_0 + \frac{2A}{R^3}\right]$$

So $-\frac{\varepsilon_2}{\varepsilon_1}\left[E_0 + \frac{2A}{R^3}\right] = -E_0 + \frac{A}{R^3}$

so $A = \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 + 2\varepsilon_2)}R^3 E_0$ and $B_1 = \frac{-3\varepsilon_2}{(\varepsilon_1 + 2\varepsilon_2)}E_0$

$$V_1 = \frac{-3\varepsilon_2}{(\varepsilon_1 + 2\varepsilon_2)}E_0 r\cos\theta \text{ and } V_2 = -\left(1 - \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 + 2\varepsilon_2)}\frac{R^3}{r^3}\right)E_0 r\cos\theta$$

Hence at $r = R$

$$\underline{\boldsymbol{E}}_1 = \frac{3\varepsilon_2}{(\varepsilon_1 + 2\varepsilon_2)}E_0\underline{\hat{k}} = E_0\underline{\hat{k}} + \frac{(\varepsilon_2 - \varepsilon_1)}{(\varepsilon_1 + 2\varepsilon_2)}E_0\underline{\hat{k}}$$

Consider Dielectric sphere in vacuum $\varepsilon_2 = \varepsilon_0$ and $\varepsilon_1 = \varepsilon_0\varepsilon_r$

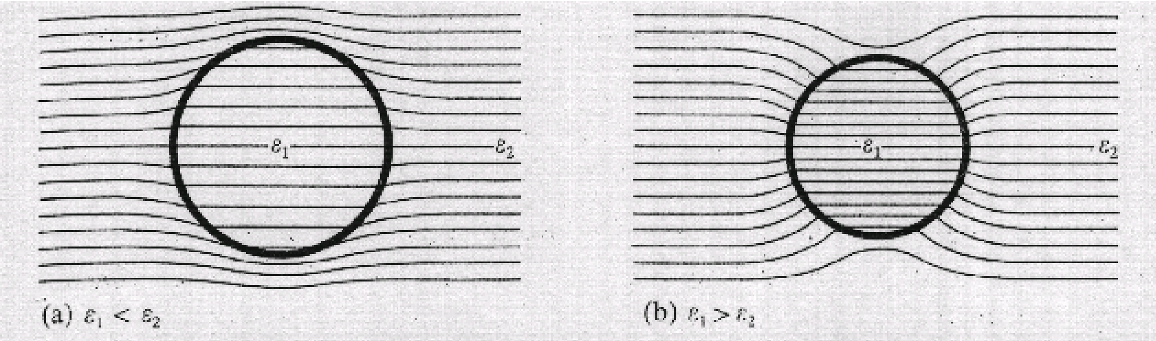
$$\therefore \underline{\boldsymbol{E}}_1 = E_0\underline{\hat{k}} + \frac{(1 - \varepsilon_r)}{(\varepsilon_r + 2)}E_0\underline{\hat{k}}$$

but $\underline{P}_1 = \varepsilon_0(\varepsilon_r - 1)E_1\underline{\hat{k}}$

$$\underline{P}_1 = \frac{3\varepsilon_0(\varepsilon_r - 1)}{(\varepsilon_r + 2)}E_0\underline{\hat{k}}$$

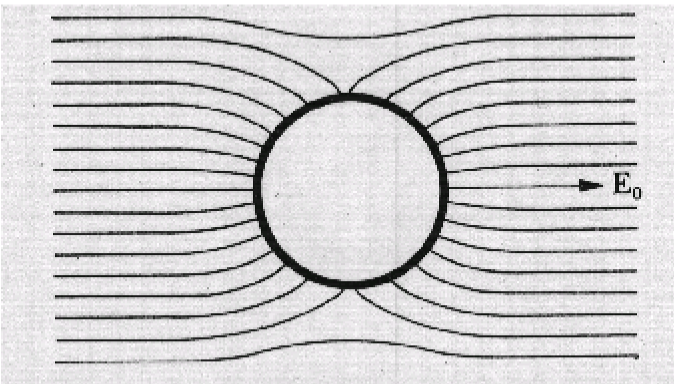
$$\therefore \qquad \underline{\boldsymbol{E}}_1 = E_0\underline{\hat{k}} - \frac{\underline{P}_1}{3\varepsilon_0}$$

DIELECTRIC SPHERE



Lines of electric displacement $\underline{\boldsymbol{D}}$ due to a dielectric sphere of relative permittivity ε_1 in a uniform electric field in a medium of relative permittivity ε_2.

CONDUCTING SPHERE



Lines of Electric Field near a conducting sphere in a uniform electric field.

See http://www.electrostatics3d.com/

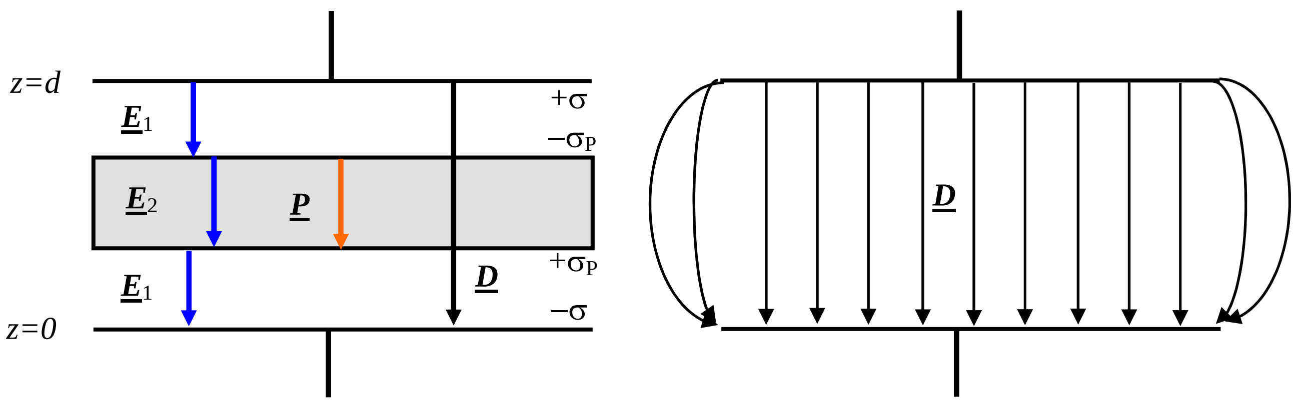
CAPACITANCE

To calculate the capacitance of any given arrangement we must calculate the P.D. between the conductors for an assumed charge.

METHOD

1. Assume charge $\pm Q$ on either conductor
2. Use GAUSS' LAW to find $\underline{\boldsymbol{D}}$ in the space between the conductors.
3. Calculate $\underline{\boldsymbol{E}}$ at each point in space using $\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}}$.
4. Find the P.D. between the conductors from $V = -\int_l \underline{\boldsymbol{E}} \bullet d\underline{\boldsymbol{l}}$ along any path joining the conductors.
5. $C = Q/V$.

EXAMPLE 1. A PARALLEL PLATE CAPACITOR.



A P.D. is applied and a charge appears on each plate $Q = \pm\sigma A$ ($\sigma =$ surface free charge density, A= area of plates).
Note there are three regions 1 and 3 are gaps between plates and dielectric where $\varepsilon_r \approx 1$ and region 2 in the dielectric ε_r is a function of position. But by integrating over one of the metal plates and using Gauss' Law we find that

$$\int_S \underline{\boldsymbol{D}} \bullet d\underline{\boldsymbol{S}} = \int_S \sigma\, dS \text{ and hence } \underline{\boldsymbol{D}} = -\sigma \underline{\hat{\boldsymbol{z}}}$$

Normal component of $\underline{\boldsymbol{D}}$ is continuous at each boundary $\therefore$ $\underline{\boldsymbol{D}}$ has the same value in all regions. Note we are ignoring the Fringing Fields i.e. assuming that linear dimensions of the plates are large compared with their separation.

In regions 1 and 3 $\varepsilon_r = 1$ so $\underline{\mathbf{E}} = \dfrac{\underline{\mathbf{D}}}{\varepsilon_0} = -\dfrac{\sigma}{\varepsilon_0}\underline{\hat{z}}$

In region 2 $\underline{\mathbf{E}} = \dfrac{\underline{\mathbf{D}}}{\varepsilon_0\varepsilon_r} = -\dfrac{\sigma}{\varepsilon_0\varepsilon_r}\underline{\hat{z}}$

Now $V = -\int_l \underline{\mathbf{E}} \bullet d\underline{\mathbf{l}} = -\int_{z=0}^{z=d} \underline{\mathbf{E}} \bullet \underline{\hat{z}} dz = \dfrac{\sigma}{\varepsilon_0}\int_{z=0}^{z=d} \dfrac{dz}{\varepsilon_r(z)}$

So $\sigma = \dfrac{\varepsilon_0 V}{\int_{z=0}^{z=d} \dfrac{dz}{\varepsilon_r(z)}}$ and $Q = \dfrac{\varepsilon_0 VA}{\int_{z=0}^{z=d} \dfrac{dz}{\varepsilon_r(z)}}$

Since $C = Q/V$ we see that $C = \dfrac{\varepsilon_0 A}{\int_{z=0}^{z=d} \dfrac{dz}{\varepsilon_r(z)}}$

Now if the dielectric is homogeneous, and fills the space between the plates (totally) $\Rightarrow$ $\varepsilon_r =$ constant and

$$C = \varepsilon_r \frac{\varepsilon_0 A}{d}$$

Remember that $\underline{\boldsymbol{D}} = \varepsilon_0 \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$ and $\underline{\boldsymbol{D}} = \varepsilon_0\varepsilon_r \underline{\boldsymbol{E}}$ so we could calculate

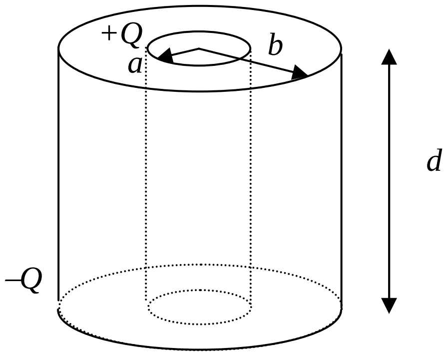
$\underline{\boldsymbol{P}}$

$$\chi = \frac{\underline{\boldsymbol{P}}}{\varepsilon_0 \underline{\boldsymbol{E}}}$$

$\rho_P = -\nabla \bullet \underline{\boldsymbol{P}}$ (= 0 for homogeneous dielectric)

$\sigma_P = \underline{\boldsymbol{P}} \bullet \underline{\hat{\boldsymbol{n}}}$

Example 2. The cylindrical capacitor (cylindrical symmetry).



Space between metal cylinders is filled with an ideal dielectric material with relative permittivity ε_r.

Construct an imaginary cylinder between the two metal coaxial cylinders and use Gauss' Law.

$$\oint_S \mathbf{D}\bullet d\mathbf{S} = D_r \times 2\pi rd = \oint_S \sigma dS = Q \quad \text{so } \mathbf{D} = D_r\hat{\mathbf{r}} = \frac{Q}{2\pi rd}\hat{\mathbf{r}} \text{ for } a < r < b$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0\varepsilon_r} = \frac{Q}{2\pi\varepsilon_0\varepsilon_r rd}\hat{\mathbf{r}}$$

$$V = -\int_{r=a}^{r=b} \mathbf{E}\bullet\hat{\mathbf{r}}dr = \frac{Q}{2\pi\varepsilon_0\varepsilon_r d}\int_{r=a}^{r=b}\frac{1}{r}\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}dr = -\frac{Q}{2\pi\varepsilon_0\varepsilon_r d}\ln\left(\frac{b}{a}\right)$$

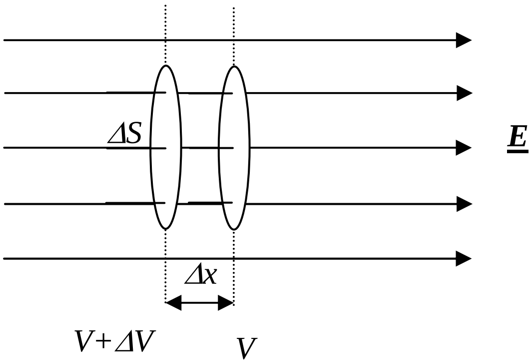
Since $C = Q/V$ we see that $C = \dfrac{2\pi d\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)}$

INTERNAL ENERGY OF A CHARGED CAPACITOR.

When a capacitor is charged the source of P.D. does work to separate the charges Q on the two conductors. This external work done by the source may be considered to reside in the field of the capacitor as potential energy.
Consider a capacitor being charged, at time t the charge on the capacitor is q. Work done by the source to increase $q \to q + dq$ requires work to be done $dW = Vdq$. Since $q = CV$ then $dW = \dfrac{qdq}{C}$ so that the W.D. to increase charge from $0 \to Q$ is

$$W.D. = \int_{q=0}^{q=Q}\frac{qdq}{C} = \frac{Q^2}{2C} = \frac{QV}{2} = \frac{CV^2}{2}$$

ENERGY DENSITY OF AN ELECTROSTATIC FIELD



Construct an imaginary surface by placing conducting plates of area ΔS on the equipotential surfaces separated by Δx. (As a conductor is an equipotential surface the presence of such plates would not disturb the field in any way).

$$\Delta C = \frac{\varepsilon_r \varepsilon_0 \Delta S}{\Delta x}$$

$$\Delta W = \frac{1}{2}\Delta C(\Delta V)^2 = \frac{1}{2}\frac{\varepsilon_r \varepsilon_0 \Delta S}{\Delta x}(\Delta V)^2 = \frac{1}{2}\varepsilon_r \varepsilon_0 \Delta S \Delta x \left(\frac{\Delta V}{\Delta x}\right)^2$$

$\underline{\boldsymbol{E}} = -\dfrac{dV}{dx}\underline{\hat{\boldsymbol{x}}}$ so as $\Delta x \to 0$ we see that $\Delta W = \dfrac{1}{2}\varepsilon_r \varepsilon_0 \Delta S \Delta x\, E^2$

$\Delta S \Delta x = \Delta \mathrm{v} =$ Volume occupied by the field in the virtual capacitor.

$\therefore$ Energy Density per unit volume

$$U = \frac{1}{2}\varepsilon_r \varepsilon_0 E^2 = \frac{1}{2}\varepsilon_r \varepsilon_0 \underline{\boldsymbol{E}} \bullet \underline{\boldsymbol{E}} = \frac{1}{2}\underline{\boldsymbol{D}} \bullet \underline{\boldsymbol{E}}$$

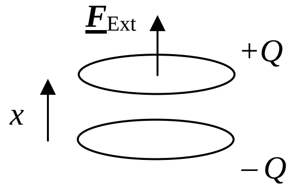
$\therefore$ Total Energy of an Electrostatic Field Occupying a volume V is

$$W = \frac{1}{2}\int_{\mathrm{v}} \underline{\boldsymbol{D}} \bullet \underline{\boldsymbol{E}}\, d\mathrm{v}$$

Note that $dU = \underline{\mathbf{E}} \bullet d\underline{\mathbf{D}} = \varepsilon_0 \underline{\mathbf{E}} \bullet d\underline{\mathbf{E}} + \underline{\mathbf{E}} \bullet d\underline{\mathbf{P}}$ where $\varepsilon_0 \underline{\mathbf{E}} \bullet d\underline{\mathbf{E}}$ is the change in energy in the absence of the dielectric and $\underline{\mathbf{E}} \bullet d\underline{\mathbf{P}}$ is the work done in polarizing the dielectric.

ELECTROSTATIC FORCES

Lines of an electric field tend to contract in a direction along the field line and to exert a sideways pressure normal to the field line ⇒ field lines will thus exert forces on their sources i.e. on the charges which give rise to them.



An external force $\underline{\boldsymbol{F}}_{Ext}$ must be applied to the plate of a capacitor (assuming the other plate is fixed), to stop them moving together since $\underline{\boldsymbol{F}}_{Int} = -\nabla W$ then $\underline{\boldsymbol{F}}_{Int} = -\underline{\boldsymbol{F}}_{Ext}$ in equilibrium

$$W = \frac{Q^2}{2C} = \frac{Q^2 x}{2\varepsilon_0 A} \quad \text{since } C = \frac{\varepsilon_0 A}{x}$$

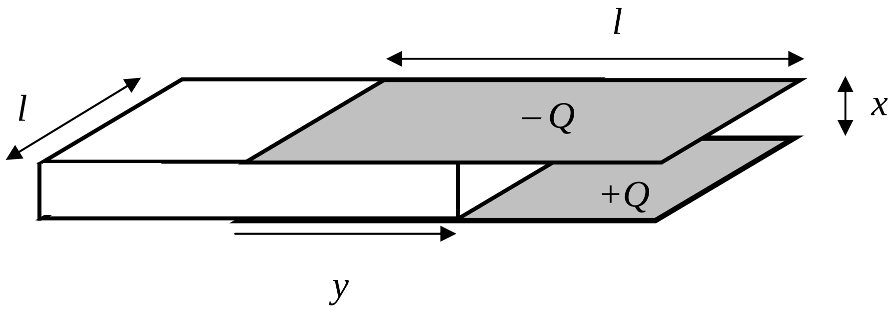
$$\underline{\boldsymbol{F}}_{Ext} = \frac{\partial W}{\partial x}\underline{\hat{\boldsymbol{x}}} = \frac{Q^2}{2\varepsilon_0 A}\underline{\hat{\boldsymbol{x}}} = \frac{C^2 V^2}{2\varepsilon_0 A}\underline{\hat{\boldsymbol{x}}} = \frac{\varepsilon_0 A V^2}{2x^2}\underline{\hat{\boldsymbol{x}}} = \frac{\varepsilon_0 A E^2}{2}\underline{\hat{\boldsymbol{x}}}$$

Field within the capacitor exerts an internal force on the plates (= and opposite to $\underline{\boldsymbol{F}}_{Ext}$) pulling the plates together. This force would compress any dielectric present and may be large enough to break the dielectric.

Note:

1. $\underline{\boldsymbol{F}} \propto E^2$ i.e. independent of direction of $\underline{\boldsymbol{E}}$.
2. For $Q =$constant from equation $W = \dfrac{Q^2}{2C}$ principle of minimization of energy ⇒ Internal forces will always act to increase capacitance.

DIELECTRIC PARTIALLY INSERTED BETWEEN CONDUCTING PLATES



$$C = \frac{\varepsilon_0 (l-y) l}{x} + \frac{\varepsilon_0 \varepsilon_r y l}{x} = \frac{\varepsilon_0 l}{x}\left[l + (\varepsilon_r - 1) y\right]$$

$$W = \frac{Q^2}{2C} = \frac{Q^2 x}{2\varepsilon_0 l\left[l + (\varepsilon_r - 1) y\right]}$$

$$\underline{\boldsymbol{F}}_x = -\frac{\partial W}{\partial x} = -\frac{Q^2}{2\varepsilon_0 l\left[l + (\varepsilon_r - 1) y\right]}\underline{\hat{\boldsymbol{x}}}$$

$$\underline{\boldsymbol{F}}_y = -\frac{\partial W}{\partial y} = \frac{(\varepsilon_r - 1) Q^2 x}{2\varepsilon_0 l\left[l + (\varepsilon_r - 1) y\right]^2}\underline{\hat{\boldsymbol{y}}}$$

So the dielectric is pulled into the gap between the plates, as well as the plates being pulled together.

MAGNETIC EFFECTS OF CURRENTS AND MAGNETOSTATICS

MAGNETIC EFFECTS OF CURRENTS

Ampere, Oested, Biot, Savart….

- Two long parallel wires carrying currents in opposite directions repel one another where as when the currents are in the same direction they attract one another.

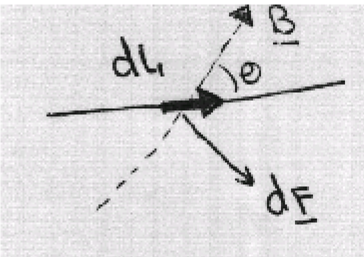
- If a wire carrying a current is placed near a magnet it experiences a force.

Current produces a magnetic field!

Introduce $\underline{\mathbf{B}}$ - MAGNETIC FLUX DENSITY [TESLA]

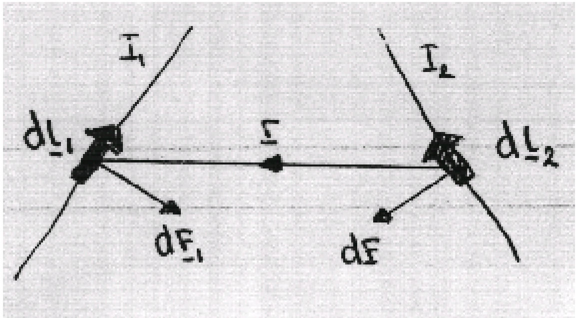
The force exerted on an element of wire $d\underline{\mathbf{l}}_1$ carrying a current I_1 at a place where the magnetic flux density $\underline{\mathbf{B}}$ can be expressed as

$$d\underline{\mathbf{F}} = I_1\left(d\underline{\mathbf{l}}_1 \times \underline{\mathbf{B}}\right) \qquad (1)$$



The force exerted on an element of wire $d\underline{\mathbf{l}}_1$ carrying a current I_1 due to another element $d\underline{\mathbf{l}}_2$ carrying a current I_2 can be expressed as

$$d\underline{\mathbf{F}}_1 = \frac{\mu_0 I_1 I_2}{4\pi r^3}\left(d\underline{\mathbf{l}}_1 \times \left(d\underline{\mathbf{l}}_2 \times \underline{\mathbf{r}}\right)\right) \qquad (2)$$



If we compare (1) and (2) we may say that the current I_2 in the element $d\underline{\mathbf{l}}_2$ produces a magnetic flux density $d\underline{\mathbf{B}}$ at a distance $\underline{\mathbf{r}}$ where

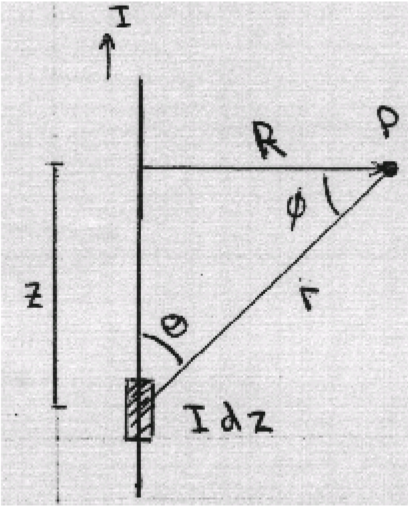
$$d\underline{\mathbf{B}} = \frac{\mu_0 I_2}{4\pi r^3}\left(d\underline{\mathbf{l}}_2 \times \underline{\mathbf{r}}\right) \tag{3}$$

$\mu_0 = 4\pi \times 10^{-7}$ NA^{-2}.

Can use (3) to calculate $\underline{\mathbf{B}}$ since

$$\underline{\mathbf{B}} = \frac{\mu_0}{4\pi}\int_l \frac{I\left(d\underline{\mathbf{l}} \times \underline{\mathbf{r}}\right)}{r^3}$$

Example 1. $\underline{\mathbf{B}}$ produced by a long straight wire carrying a current I.



Magnetic Flux at point P due to element dz is

$$d\underline{\mathbf{B}}_P = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}z\,r\sin\theta}{r^3}$$

[Directed into the page]

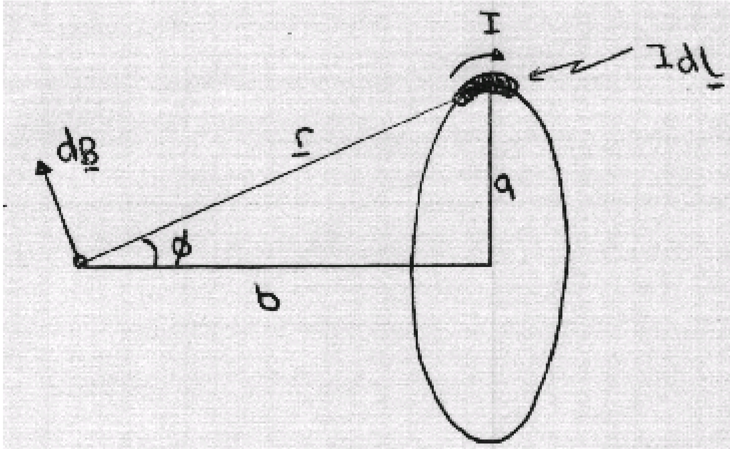
$z = R\tan\phi$ $\sin\theta = \cos\phi$

$\mathrm{d}z = R\sec^2\phi\,\mathrm{d}\phi$ $R = r\cos\phi$

Therefore $d\underline{\mathbf{B}}_P = \dfrac{\mu_0}{4\pi}\dfrac{I\cos^2\phi\,\mathrm{R}\sec^2\phi\,\mathrm{d}\phi\cos\phi}{R^2}$

$$\underline{\mathbf{B}}_P = \frac{\mu_0 I}{4\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos\phi\,\mathrm{d}\phi = \frac{\mu_0 I}{2\pi R}$$

Example 2. $\underline{\mathbf{B}}$ along the axis of a current loop I.



$I\,\mathrm{d}\underline{\mathbf{l}} \perp \underline{\mathbf{r}}$. Therefore

$$\mathrm{d}\underline{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}l}{r^2}$$

Must sum results from around whole loop – each component contributes $\mathrm{d}\underline{\mathbf{B}} = \mathrm{dB}\sin\phi$ along the axis and the $\perp$ components cancel.

$$\underline{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\int_l \frac{\sin\phi\,\mathrm{d}l}{r^2} = \frac{\mu_0 I \sin\phi}{4\pi r^2}\int_l \mathrm{d}l = \frac{\mu_0 I \sin\phi}{4\pi r^2}2\pi a$$

$$\sin\phi = \frac{a}{\left(a^2+b^2\right)^{1/2}} \qquad r = \left(a^2+b^2\right)^{1/2}$$

So $\underline{\mathbf{B}} = \dfrac{\mu_0 I}{2}\dfrac{a^2}{\left(a^2+b^2\right)^{3/2}}$

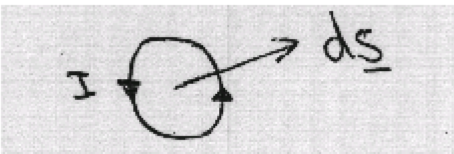
When $b >> a$ then $\underline{\mathbf{B}} = \dfrac{\mu_0 I}{2\pi}\dfrac{a\pi^2}{r^3} = \dfrac{\mu_0}{2\pi r^3}\left(AI\right) = \dfrac{\mu_0\, m}{2\pi r^3}$

$A =$ Area of loop and the magnetic dipole moment $m = IA\,[\mathrm{Am}^2]$

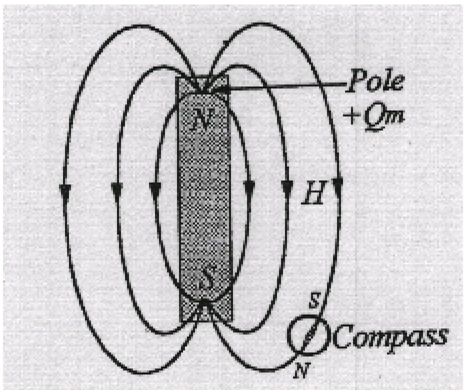
AMPERES MAGNETIC DIPOLE

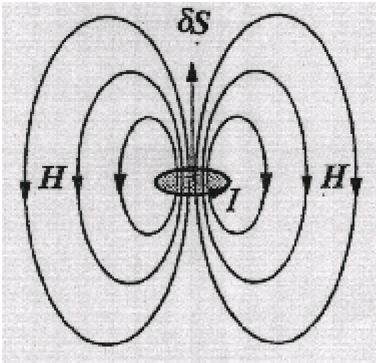
Ampere noted that the magnetic field configuration produced by a small loop of current is identical to that produced by a small permanent magnet

Convention I circumscribes the vector $\mathrm{d}\underline{\mathbf{S}}$ in a right hand sense and $\underline{\mathbf{m}} = I\mathrm{d}\underline{\mathbf{S}}$



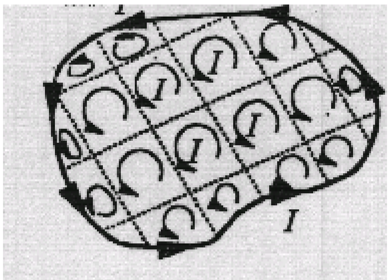
Origin of all magnetism is electrical currents. Ampere proposed that permanent magnetism was the result of ‘Atomic currents’ i.e. electrical currents flowing at the atomic level.





Note $\underline{\mathbf{H}}$ in diagram not $\underline{\mathbf{B}}$ we will shortly find out how $\underline{\mathbf{H}}$ is related to $\underline{\mathbf{B}}$

MAGNETIC EFFECT OF A CURRENT LOOP OF ANY SIZE



Can subdivide large circuit into a network of small circuits in each of which a current I circulates. In the interior the current in adjacent loops cancel $\Rightarrow$ Left with current flowing around periphery.

AMPERES CIRCUITAL LAW IN VACUUM

Have shown that for a long wire carrying a current I at a distance r, $\underline{\mathbf{B}} = \dfrac{\mu_0 I}{2\pi R}$ and that the lines of $\underline{\mathbf{B}}$ are concentric around the wire.

If we perform a line integral $\oint \underline{\mathbf{B}} \bullet \mathrm{d}\underline{\mathbf{l}}$ on a closed path $\underline{\mathbf{l}}$ which forms a circular loop (radius r) around the wire we get

$$\oint \underline{\mathbf{B}} \bullet \mathrm{d}\underline{\mathbf{l}} = \frac{\mu_0 I}{2\pi r}\oint \mathrm{dl} = \frac{\mu_0 I}{2\pi r} 2\pi r = \mu_0 I$$

i.e. $\oint \underline{\mathbf{B}} \bullet \mathrm{d}\underline{\mathbf{l}} = \mu_0 I$

This works for any path! Why?

Because any given closed path around the conductor is approximated by segments that are either radial or circular arcs about the conductor. The contribution of radial segments to $\oint \underline{\mathbf{B}} \bullet \mathrm{d}\underline{\mathbf{l}}$ is zero since everywhere $\underline{\mathbf{B}}$ $\perp$ to the radius vector $\therefore$ $\oint \underline{\mathbf{B}} \bullet \mathrm{d}\underline{\mathbf{l}}$ is the value over only the circular segments.

In fact for any current threading the area enclosed by a chosen path

$$\oint \underline{\mathbf{B}} \bullet \mathrm{d}\underline{\mathbf{l}} = \mu_0 I$$

$\underline{\mathbf{B}}$ AND $\underline{\mathbf{H}}$

In a vacuum $\underline{\mathbf{B}} = \mu_0 \underline{\mathbf{H}}$

$\underline{\mathbf{B}} =$ Magnetic flux density of the magnetic induction [Tesla=NA^{-1}m^{-1}]

$\underline{\mathbf{H}} =$ Magnetic Field [Am^{-1}]

$\mu_0 = 4\pi \times 10^{-7}$ [NA^{-2} = Tesla A^{-1}m^{-1}] and is called the PERMEABILITY OF FREE SPACE.

In vacuum

$\underline{\mathbf{B}} = \mu_0 \underline{\mathbf{H}} \Rightarrow \qquad \oint \underline{\mathbf{H}} \bullet \mathrm{d}\underline{\mathbf{l}} = I$ (A)

Amperes Circuital Law

and $\qquad \underline{\mathbf{H}} = \int_l \dfrac{I\left(d\underline{\mathbf{l}} \times \underline{\mathbf{r}}\right)}{4\pi r^3}$ (B)

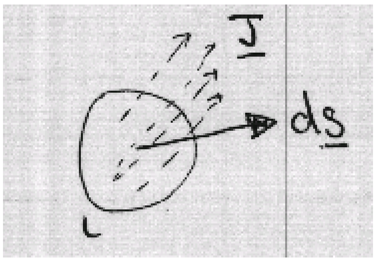
[$\underline{\mathbf{H}}$ produced at a distance $\underline{\mathbf{r}}$ from I flowing along path $\underline{\mathbf{l}}$]

Both (A) and (B) are true in any media.

Note: since $\oint \underline{\mathbf{H}} \bullet \mathrm{d}\underline{\mathbf{l}} = I$ $\underline{\mathbf{H}}$ field is not conservative unless $I = 0$.

CURRENT DENSITY AND AMPERES LAW

If a path $\underline{\mathbf{l}}$ is drawn within a current distribution, the total current I linked by the path is $I = \int_S \underline{\mathbf{J}} \bullet \mathrm{d}\underline{\mathbf{S}}$



$\underline{\mathbf{J}} =$ Current density [Am^{-2}]

Since $\oint \underline{\mathbf{H}} \bullet \mathrm{d}\underline{\mathbf{l}} = I = \int_S \underline{\mathbf{J}} \bullet \mathrm{d}\underline{\mathbf{S}}$

And Stokes Theorem states that $\oint_l \underline{\mathbf{H}} \bullet \mathrm{d}\underline{\mathbf{l}} = \int_S \nabla \times \underline{\mathbf{H}} \bullet \mathrm{d}\underline{\mathbf{S}} = \int_S \underline{\mathbf{J}} \bullet \mathrm{d}\underline{\mathbf{S}}$

$\Rightarrow \qquad \nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}}$

Differential form of Amperes Law (only true for constant I, $\underline{\mathbf{H}}$ and $\underline{\mathbf{J}}$).

GAUSS, LAW IN MAGNETISM

Magnetic Flux = Total lines of $\underline{\mathbf{B}}$ through a given area.

$$\Phi = \int_S \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}}$$

Remember $\oint_S \underline{\mathbf{E}} \bullet d\underline{\mathbf{S}} = \dfrac{q}{\varepsilon_0}$ so in the absence of any charge $\oint_S \underline{\mathbf{E}} \bullet d\underline{\mathbf{S}} = 0$

Magnetic Monopoles don't exist – only magnetic dipoles

$\Rightarrow \oint_S \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}} = 0$ always. Since Gauss' Theorem states $\int_v \nabla \bullet \underline{\mathbf{B}} dv = \oint_S \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}}$

$$\oint_S \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}} = 0 \text{ means that } \nabla \bullet \underline{\mathbf{B}} = 0 \text{ always.}$$

Lines of $\underline{\mathbf{B}}$ always form closed paths. No sources of $\underline{\mathbf{B}}$.

MAGNETOSTATICS
No charges, no electrical fields. Steady currents and time independent magnetic field.

$$\oint_l \underline{\mathbf{H}} \bullet d\underline{\mathbf{l}} = I$$

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}}$$

$$\nabla \bullet \underline{\mathbf{B}} = 0$$

Only if $\underline{\mathbf{J}} = 0$ can we define a magnetic scale potential ϕ_M such that

$$\underline{\mathbf{H}} = -\nabla \phi_M$$

ELECTROSTATICS
$\underline{\mathbf{B}}, \underline{\mathbf{H}}$ and $\underline{\mathbf{J}}$ all zero. $\underline{\mathbf{D}}$ and $\underline{\mathbf{E}}$ time independent.

$$\oint_l \underline{\mathbf{E}} \bullet d\underline{\mathbf{l}} = 0$$

$$\nabla \times \underline{\mathbf{E}} = 0$$

$$\nabla \bullet \underline{\mathbf{D}} = \rho_f$$

$$\underline{\mathbf{E}} = -\nabla V$$

Note: $\nabla \times \underline{\mathbf{H}} = \nabla \times \left(-\nabla \phi_M\right) = 0$

THE MAGNETIC PROPERTIES OF MATERIALS

All magnetic materials are affected by the presence of a magnetic field. When a magnetic field of strength $\underline{\mathbf{H}}$ exists within a substance it permeates the material and produces "induced" magnetic dipole throughout the body of the material. The macroscopic measure of this effect is the "MAGNETISATION" $\underline{\mathbf{M}}$.

$\underline{\mathbf{M}}$ IS THE INDUCED MAGNETIC DIPOLE MOMENT PER UNIT VOLUME [$Am^2m^{-3} \equiv Am^{-1}$] – same units as $\underline{\mathbf{H}}$.

$\underline{\mathbf{M}}$ is the magnetic equivalent of the polarisation $\underline{\mathbf{P}}$ in electrostatics.

For simple magnetic media which are linear ($\underline{\mathbf{M}} \propto \underline{\mathbf{H}}$), homogeneous and isotropic then

$$\underline{\mathbf{M}} = \chi_M \underline{\mathbf{H}}$$

where $\underline{\mathbf{H}}$ is the field strength within the medium and χ_M is the MAGNETIC SUSCEPTIBILITY [Dimensionless].

[In electrostatics $\underline{\mathbf{P}} = \varepsilon_0 \chi \underline{\mathbf{E}}$ where χ is the ELECTRIC SUSCEPTRIBILITY]

At room temperature the magnetic susceptibility is typically small and independent of $\underline{\mathbf{H}}$ - BUT for FERROMAGNETIC materials χ_M is large and very dependent on $\underline{\mathbf{H}}$.

DIAMAGNETISM (χ_D)**,**

PARAMAGNETSIM (χ_P)**,**

and FERROMAGNETISM (χ_F)

DIAMAGNETISM

DIAMAGNETIC substances are composed of atoms (or molecules) that have no permanent magnetic moment. The atom consists of closed shells, so that the magnetic moments associated with individual electron orbitals cancel out and the total angular momentum quantum number $J = 0$.

It can be shown that $\chi_D = -\dfrac{n\mu_0 e^2 Z\langle r^2\rangle}{6m_e}$ where n is the number of atoms per unit volume, Z is the number of electrons on each atom, $\langle r^2\rangle$ is the average radius of the electron orbital and all other terms have their usual meaning. Note the minus sign, the "INDUCED DIPOLE MOMENT" (or induced current) opposes the applied magnetic flux/field, χ_D is independent of temperature and small in magnitude. Typically $\chi_D \approx -5\times10^{-10} Z$ for the noble gases (He, Ne Ar, Kr, Xe) with 2.69×10^{25} atoms m^{-3} at RTP.

PARAMAGNETISM

PARAMAGNETIC substances consist of atoms, ion or molecules that possess a permanent magnetic dipole moment. This atomic electron dipole moment arises from the orbital motion of the electron and the electron spin.

The electron magnetic moment of a free atom can be expressed as

$$\underline{\boldsymbol{\mu}} = g_J \mu_B \underline{\mathbf{J}}$$

where $g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$ is the Lande g-factor, $\mu_B = 9.27\times10^{-24}$ Am^{-2} is the Bohr Magneton, and $\underline{\mathbf{J}} = \underline{\mathbf{L}} + \underline{\mathbf{S}}$ is the "Effective spin" angular momentum. In the absence of an applied magnetic field the directions of the magnetic dipole moments ($\underline{\boldsymbol{\mu}}$) of the individual atoms are randomised by thermal energy and the net magnetic moment of a macroscopic volume is zero. When $\underline{\mathbf{B}}$ is applied dipole tend to align themselves in the direction of the field – Magnetic alignment energy $= -\underline{\boldsymbol{\mu}} \bullet \underline{\mathbf{B}}$. If $\left|\underline{\boldsymbol{\mu}} \bullet \underline{\mathbf{B}}\right| << k_B T$ then the result is a small net alignment in the direction of the field – induced magnetic moment is in the same direction as the applied $\underline{\mathbf{B}} \Rightarrow \chi_P > 0$ (positive!).

It the atoms/molecules/ions are sufficiently far apart that their mutual interactions can be neglected (i.e. gas of low concentration of paramagnetic ions in a diamagnetic solid) then

$$\chi_P = \frac{\mu_0 n \mu^2}{3k_B T} = \frac{C}{T} \qquad \text{if } \left|\underline{\boldsymbol{\mu}} \bullet \underline{\mathbf{B}}\right| << k_B T$$

$n =$ Number of atoms/molecules/ions per unit volume

$\mu^2 = g_J^2 \mu_B^2 J(J+1)$

$C =$ Curie Constant

Note χ_P is positive, small and depends on temperature as $1/T$.

For solids and liquids where interactions between paramagnetic atoms/ions cannot be neglected

$$\chi_P \approx \frac{C}{T-\theta} \qquad \text{Only works for } T > |\theta|$$

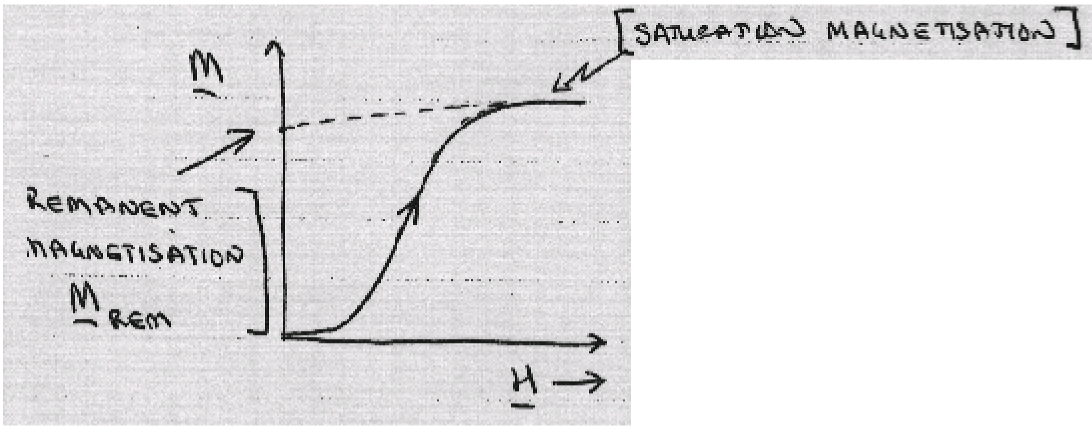
$C =$ Constant, $\theta =$Weiss constant can be positive or negative.

FERROMAGNETSIM

Ferromagnetic substances are all solid, and each is characterised by a certain temperature known as the CURIE POINT at which the properties change abruptly.

- Magnetisation is not proportional to $\underline{\mathbf{H}}$, in certain situations a susceptibility of several thousand an be measured and very large magnetisations can be achieved.
- The value of the magnetisation depends not only on the applied field but also on the previous history of the samples.
- A sample may retain its magnetisation even in the absence of an external applied field – PERMANENT MAGNETS. However, it is notable that the very same material can also exist is a state showing little or no permanent magnetism.

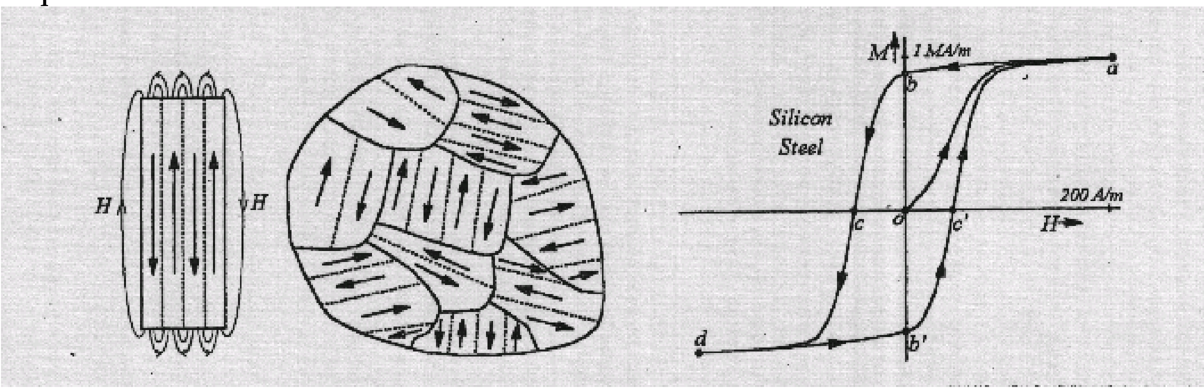
The ultimate source of magnetic moments in ferromagnetic materials turns out to be the magnetic moments arising from electron spin – the big difference in Ferromagnetics (cf. Paramagnetism) is that there are large interactions between spins that cause them to align parallel with each other – even at room temperature thermal vibrations cannot destroy the alignment.



Initially un-magnetised samples – as $\underline{\mathbf{H}}$ increases $\underline{\mathbf{M}}$ increases and eventually saturates. If $\underline{\mathbf{H}}$ is then decreases $\underline{\mathbf{M}}$ does not go back to zero!

Ferromagnetic “Weiss” Domains
Interaction between spins results in preferential alignment – a quantum cooperative phenomenon! So why aren’t a lumps of iron spontaneously magnetised? A magnetic field outside the material involves stored energy

$= \frac{1}{2}\int_v \underline{\mathbf{H}} \bullet \underline{\mathbf{B}}\,dv$ [J]. If the sample is "broken up" into differently oriented "domains" the stored energy in this field is greatly reduced. This decrease must be balanced against the energy stored in making domain walls. Now, look at the magnetisation curve. As $\underline{\mathbf{H}}$ increases at first nothing much happens, but then preferentially oriented grains start to grow rapidly at the expense of others. Finally the magnetisation increases slowly as the non-preferentially oriented domains rotate parallel to the applied field (saturation). The whole process requires Domain Wall motion.



Hysteresis: If the applied field is now reduced $\underline{\mathbf{M}}$ does not follow the same path (hysteresis).

The Curie Temperature (T_c)**.** Heating a ferromagnetic material above T_c causes a transition to the PARAMAGNETIC STATE, the susceptibility can decrease by many orders of magnitude.

Relation between $\underline{\mathbf{B}}$, $\underline{\mathbf{H}}$, and $\underline{\mathbf{M}}$.

$$\underline{\mathbf{B}} = \mu_0(\underline{\mathbf{H}} + \underline{\mathbf{M}}) \quad \text{[Tesla]}$$

For linear media $\underline{\mathbf{M}} = \chi_M \underline{\mathbf{H}} \quad \Rightarrow \quad \underline{\mathbf{B}} = \mu_0(1+\chi_M)\underline{\mathbf{H}}$

Or $\underline{\mathbf{B}} = \mu_0 \mu_r \underline{\mathbf{H}}$ where $\mu_r = 1 + \chi_M$

$\mu_r =$ RELATIVE PERMEABILITY [Dimensionless]

SUMMARY

DIAMAGNETIC MATERIALS: $|\chi_M| << 1$ and NEGATIVE $\mu_r \le 1$

PARAMGNETIC MATERIALS: $|\chi_M| << 1$ and POSITIVE $\mu_r \ge 1$

FEROMAGNETIC MATERIALS: $|\chi_M| >> 1$ and POSITIVE $\mu_r \approx 10 - 10{,}000$

BOUNDARY CONDITIONS IN MAGNETISM

We will consider boundaries between linear, isotropic and homogeneous media.

☺ THE TANGENTIAL COMPONENT OF **H** IS CONTINUOUS ACROSS A BOUNDARY PROVIDED THAT THERE IS NO SURFACE CURRENT ON THE BOUNDARY.

☺ THE NORMAL COMPONENT OF **B** IS CONTINUOUS ACROSS A BOUNDARY.

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ANY SOLUTION TO AN MAGNETOSTATICS PROBLEM MUST SATISFY THE BOUNDARY CONDITIONS.
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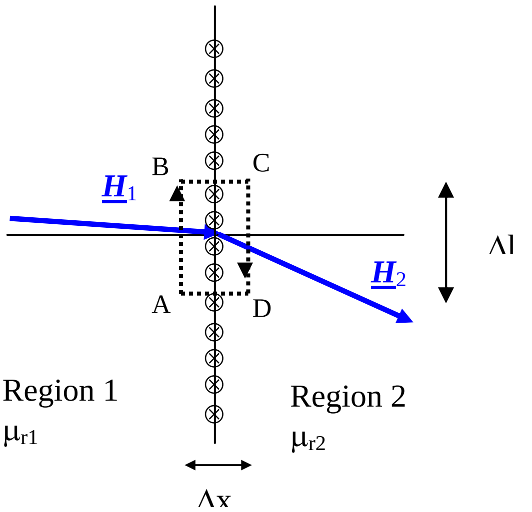
THE TANGENTIAL COMPONENT OF **H**



⊗ : Current flowing along the surface between regions 1 and 2 (ie. Into the page).

In region 1 the magnetic field ($\underline{\mathbf{H}}_1$) makes an angle of θ_1 with the normal to the interface, and in region 2 $\underline{\mathbf{H}}_2$ makes an angle of θ_2 with the normal to the interface. Therefore the magnitudes of the components of $\underline{\mathbf{H}}$ tangential to the interface are $H_{1t} = H_1 \sin\theta_1$ and $H_{2t} = H_2 \sin\theta_2$.

The line integral of $\underline{\mathbf{H}}$ around any closed path in a magnetostatic field is equal to the current threading the path $\oint_l \underline{\mathbf{H}} \bullet d\underline{\mathbf{l}} = \mathrm{I} = \int_S \underline{\mathbf{J}} \bullet d\underline{\mathbf{S}}$. So if we consider the path ABCD where AB, CD = Δl and BC, DA $= \Delta x$, in the limit $\Delta x \to 0$

$\int_S \underline{\mathbf{J}} \bullet d\underline{\mathbf{S}} \rightarrow 0$ since current density is finite in everything except a perfect conductor.

$$\oint_l \underline{\mathbf{H}} \bullet d\underline{\mathbf{l}} = H_{1t}\Delta l - H_{2t}\Delta l = 0$$

$\therefore$ $$H_{1t} - H_{2t} = 0$$

then $$H_{1t} = H_{2t}$$

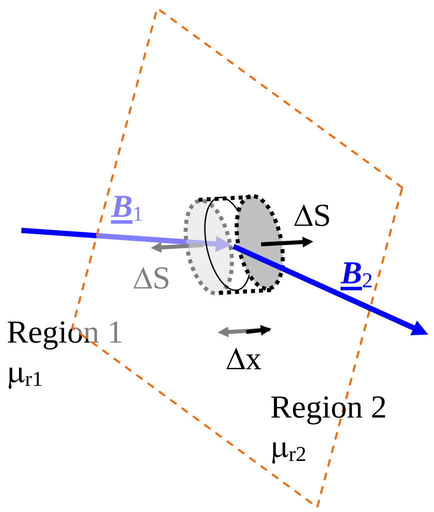
(Tangential component of $\underline{\mathbf{H}}$ is continuous across a boundary)

Because $\underline{\mathbf{B}} = \mu_0 \mu_r \underline{\mathbf{H}}$ $$\frac{B_{2t}}{\mu_2} = \frac{B_{1t}}{\mu_1}$$

i.e. The tangential component of $\underline{\mathbf{B}}$ is discontinuous as the boundary.

(Aside: For a perfect conductor can consider a surface charge per unit length j_S flowing in a vanishing thin layer at the interface, then the boundary condition becomes $H_{1t} - H_{2t} = j_S$)

THE NORMAL COMPONENT OF $\underline{\mathbf{B}}$



Construct a Gaussian surface in the form of a cylinder that "straddles" the boundary.

Make the thickness of the box $\Delta x \rightarrow 0$ so that no lines of $\underline{\mathbf{B}}$ come out of the sides of the little Gaussian cylinder.

In region 1 the electric displacement $\underline{\mathbf{B}}_1$ makes an angle of θ_1 with the normal to the interface, and in region 2 $\underline{\mathbf{B}}_2$ makes an angle of θ_2 with the normal to the interface. Therefore the magnitudes of the components of $\underline{\mathbf{B}}$ normal to the interface are $B_{1n} = B_1 \cos\theta_1$ and $B_{2n} = B_2 \cos\theta_2$.

Since $\oint_S \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}} = 0$ then $\oint_{S1} \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}} = -\oint_{S2} \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}}$

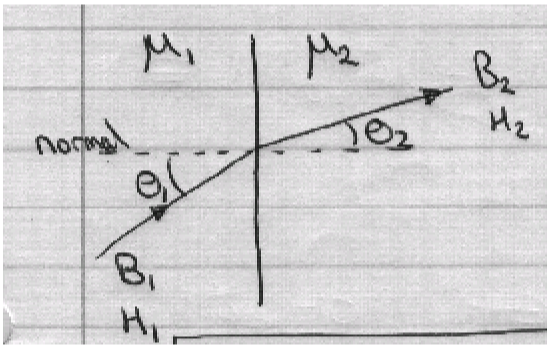
$$B_{2n}\Delta S = B_{1n}\Delta S$$

$$\therefore \qquad B_{1n} = B_{2n}$$

Because $\underline{\mathbf{B}} = \mu_0\mu_r \underline{\mathbf{H}}$ there is a "jump" in the normal component of the magnetic field at the boundary.

$$\mu_2 H_{2n} = \mu_1 H_{1n}$$

REFRACTION OF LINES OF $\underline{\mathbf{B}}$ AND $\underline{\mathbf{H}}$.



Boundary conditions:

$j_S = 0 \qquad H_{1t} = H_{2t}$ or

$j_S = 0 \; H_1 \sin\theta_1 = H_2 \sin\theta_2$

and

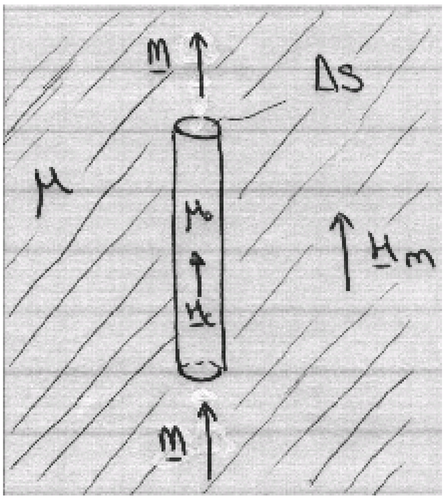
$B_{2n} = B_{1n}$ or $B_1 \cos\theta_1 = B_2 \cos\theta_2$

So $\dfrac{B_1}{H_1}\cot\theta_1 = \dfrac{B_2}{H_2}\cot\theta_2$ and since $\underline{\mathbf{B}} = \mu_0\mu_r \underline{\mathbf{H}}$ we find that

$\mu_1 \cot\theta_1 = \mu_2 \cot\theta_2$: Refraction formula for magnetic field lines.

FIELDS WITHIN CAVITIES IN A MEDIUM

(OR RODS AND DISCS OF MAGNETIC MATERIAL IN A PRE-EXISTING FREE SPACE $\underline{\mathbf{B}}_0$ AND $\underline{\mathbf{H}}_0$)

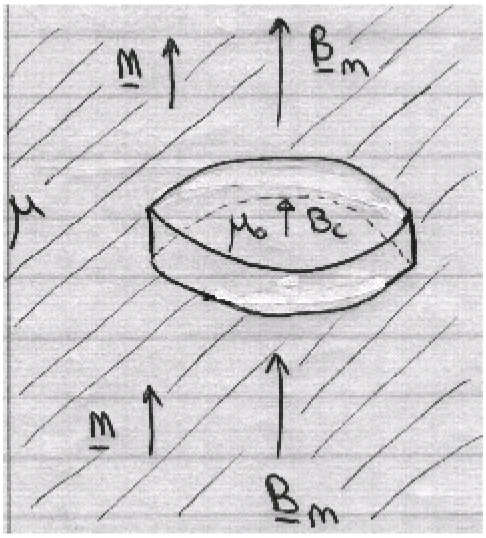


A NEEDLE SHAPED CAVITY

Pre-existing magnetic field in media $\underline{\mathbf{H}}_M$. If cavity long and thin so that we can ignore the ends (stay away from ends!)

$\underline{\mathbf{H}}_C = \underline{\mathbf{H}}_M$ [Think H$_{\text{Tangential}}$!] and $\dfrac{\underline{\mathbf{B}}_C}{\mu_0} = \dfrac{\underline{\mathbf{B}}_M}{\mu}$

DISC-SHAPED CAVITY



If cavity short and wide so that we can ignore the edges (stay away from the edges!)

$\underline{\mathbf{B}}_C = \underline{\mathbf{B}}_M$ [Think B_{Normal} !]

$$\mu_0 \underline{\mathbf{H}}_C = \mu \underline{\mathbf{H}}_M$$

i.e. $\underline{\mathbf{H}}_C$ is $\dfrac{\mu}{\mu_0} = \mu_r$ times its value in the medium

MAGNETIC CIRCUITS

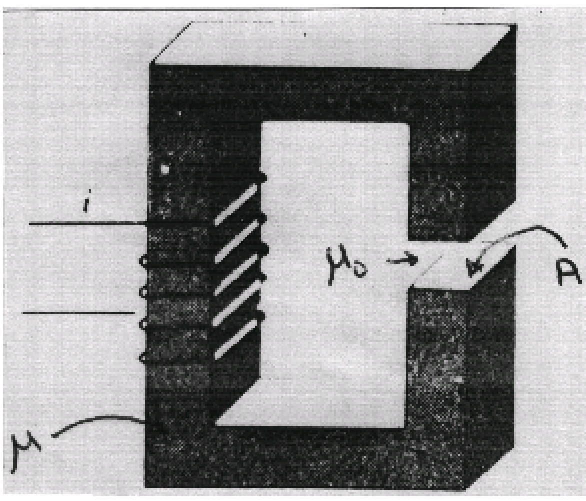
General problem of magnetic bodies in external fields is extremely difficult! We are involved in the simultaneous solution of

A: $\oint_l \underline{\mathbf{H}} \bullet d\underline{\mathbf{l}} = \mathrm{I}$ B: $\oint_S \underline{\mathbf{B}} \bullet \mathrm{d}\underline{\mathbf{S}} = 0$ C: $\underline{\mathbf{B}} = \mu_0 \mu_r \underline{\mathbf{H}}$

and the boundary conditions for $\underline{\mathbf{B}}$ and $\underline{\mathbf{H}}$.

Don't panic! There is one kind of situation involving Ferromagnetic materials that is practically important and easy to solve (approximately).

THE ELECTROMAGNET: What is $\underline{\mathbf{H}}$ in the air gap?



We know that the current (i) in the coil, and the number of turns N, the cross sectional area is A and the value of μ for all parts. We assume that the lines of $\underline{\mathbf{B}}$ are parallel to and confined within the surface of the ferromagnetic (good approximation when μ large).

Note everywhere
$\underline{\mathbf{H}}$ parallel to path

$$A \Rightarrow \oint_l \underline{\mathbf{H}} \bullet d\underline{\mathbf{l}} = \mathrm{H}_1 l_1 + \mathrm{H}_2 l_2 + \mathrm{H}_3 l_3 + \mathrm{H}_4 l_4 + \mathrm{H}_5 l_5 + \mathrm{H}_6 l_6$$
$$= \mathrm{H}_\mathrm{M}\left(l_1 + l_2 + l_3 + l_5 + l_6\right) l_1 + \mathrm{H}_0 l_4$$
$$= N\,i$$

$$B \Rightarrow \oint_S \underline{\mathbf{B}} \bullet \mathrm{d}\underline{\mathbf{S}} = 0 \qquad \Rightarrow B_0 A = B_M A$$

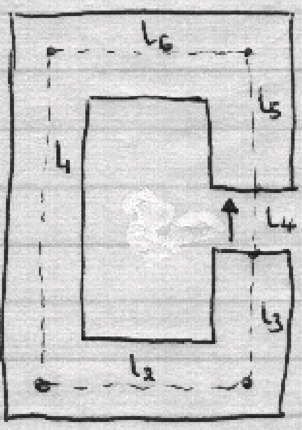
[Or think of $\mathrm{B}_{\text{normal}}$ at the ferromagnet-free space interface. This must be continuous]

$$C \Rightarrow \underline{\mathbf{B}} = \mu_0 \mu_r \underline{\mathbf{H}} \qquad \Rightarrow \mu_0 H_0 = \mu_M H_M$$

$$\therefore \frac{\mu_0}{\mu_\mathrm{M}} \mathrm{H}_0 \left(L - l_4\right) + \mathrm{H}_0 l_4 = Ni$$

$$\mathrm{H}_0 = \frac{Ni}{\frac{\mu_0}{\mu_\mathrm{M}}\left(L - l_4\right) + l_4}$$ and since $\mu_M = \mu_r \mu_0$

$$\mathrm{H}_0 = \frac{\mu_r Ni}{L + \left(\mu_r - 1\right) l_4}$$ and $$\mathrm{B}_0 = \frac{\mu_0 \mu_r Ni}{L + \left(\mu_r - 1\right) l_4}$$



$L = l_1 + l_2 + l_3 + l_4 + l_5 + l_6$

MAXWELLS EQUATIONS

GAUSS' LAW

(i) ELECTROSTATICS

$\oint_S \underline{\mathbf{D}} \bullet d\underline{\mathbf{S}} = Q = \int_{\mathrm{v}} \rho \, d\mathrm{v}$ or $\nabla \bullet \underline{\mathbf{D}} = \rho$

$\underline{\mathbf{D}} =$ Electric Displacement [Cm^{-2}]

(i) MAGNETOSTATICS

$\oint_S \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}} = 0$ or $\nabla \bullet \underline{\mathbf{B}} = 0$

$\underline{\mathbf{B}} =$ Magnetic Flux Density [Tesla]

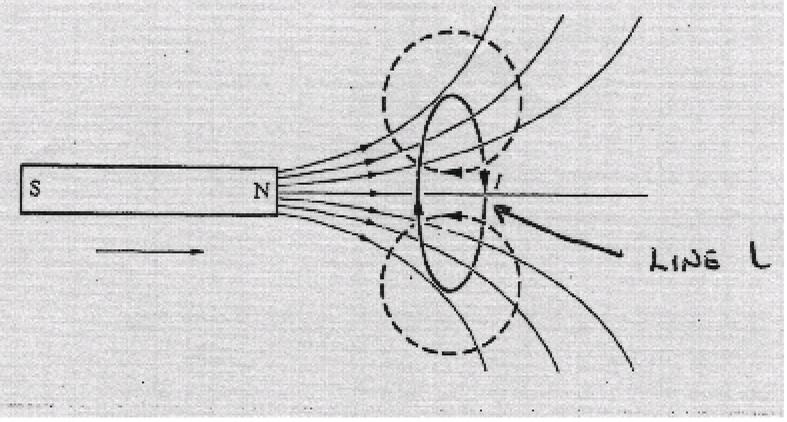
AMPERES CIRCUITAL LAW

$\oint_l \underline{\mathbf{H}} \bullet \mathrm{d}\underline{\mathbf{l}} = I = \int_S \underline{\mathbf{J}} \bullet \mathrm{d}\underline{\mathbf{S}}$ or $\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}}$

$\underline{\mathbf{H}} =$ Magnetic Field [Am^{-1}]
$\underline{\mathbf{J}} =$ Current density [Am^{-2}]

FARADAY LAW OF ELECTROMAGNETIC INDUCTION

Oestred showed that an electrical current produces a magnetic field (1820). 1831 $\Rightarrow$ FARADAY found that a current was induced in a circuit when a magnetic field that links the circuit changes.



The EMF induced in a circuit (given by line l) is $\varepsilon_V = -\frac{\partial \Phi}{\partial t}$ (minus sign comes from Lenz's Law).

$\Phi = \int_S \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}}$ (Any surface whose boundary is the line l)

$\Phi =$ MAGNETIC FLUX linked by the circuit [Tesla m^2 or Weber, Wb]

The induced EMF ε_V is equal the line integral of the induced $\underline{\mathbf{E}}$ [Vm^{-1}] electric field around the coil.

$$\oint_l \underline{\mathbf{E}} \bullet \mathrm{d}\underline{\mathbf{l}} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}\int_S \underline{\mathbf{B}} \bullet d\underline{\mathbf{S}}$$

Using Stokes Theorem $\oint_l \underline{\mathbf{E}} \bullet \mathrm{d}\underline{\mathbf{l}} = \int_S \nabla \times \underline{\mathbf{E}} \bullet \mathrm{d}\underline{\mathbf{S}}$

$$\int_S \nabla \times \underline{\mathbf{E}} \bullet \mathrm{d}\underline{\mathbf{S}} = -\int_S \frac{\partial \underline{\mathbf{B}}}{\partial t} \bullet d\underline{\mathbf{S}}$$

$$\therefore \ \nabla \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

<u>CONSTITUTIVE RELATIONS</u>

Ohms Law $V = IR, R = \frac{\rho_R l}{A}$ $\rho_R =$ Resistivity [Ωm]

$\sigma_C = \frac{1}{\rho_R}$ $\sigma_C =$ Conductivity [Ω^{-1}m^{-1}]

$I = \frac{V}{R} = V\frac{A}{\rho_R l} = \frac{V}{l}\sigma_C A$, re-arrange and we get $\frac{I}{A} = J = \sigma_C E$

Or in vector form (Homogeneous, isotropic media) $\underline{\mathbf{J}} = \sigma_C \underline{\mathbf{E}}$

So we now have:

$\underline{\boldsymbol{D}} = \varepsilon_0 \underline{\boldsymbol{E}} + \underline{\boldsymbol{P}}$ $\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}}$

$\underline{\mathbf{B}} = \mu_0 (\underline{\mathbf{H}} + \underline{\mathbf{M}})$ $\underline{\mathbf{B}} = \mu_0 \mu_r \underline{\mathbf{H}}$

$\underline{\mathbf{J}} = \sigma_C \underline{\mathbf{E}}$

POWER DISSIPATION AND JOULE HEATING

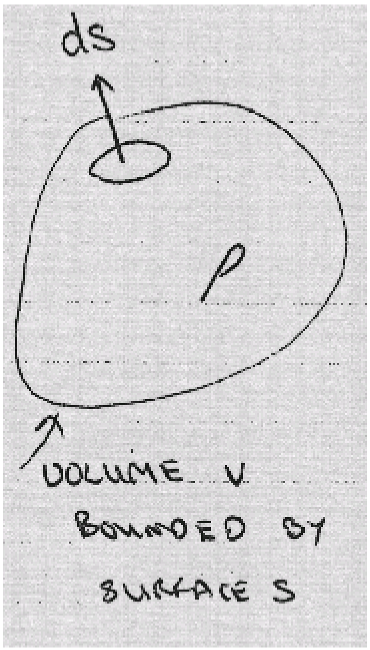
Power is dissipated in the resistance R causing "Joule Heating".

$$W = IV = \frac{V^2}{R} = I^2 R$$

$$W = J^2 A^2 \frac{l}{\sigma_C A} = \frac{\sigma_C EJ}{\sigma_C} Al = EJ \times [Volume]$$

$$W = \int_v \underline{\mathbf{J}} \bullet \underline{\mathbf{E}}\,\mathrm{dv}$$ [Now works if $\underline{\mathbf{E}}$ and $\underline{\mathbf{J}}$ in different directions and/or vary with position]

THE EQUATION OF CONTINUITY



Imagine a volume of space v that at a given time contains a total charge Q, where

$$Q = \int_v \rho\,\mathrm{dv}$$

If charge can flow out (or into) the volume then there is a current.

$$I = -\frac{\partial Q}{\partial t} = -\int_v \frac{\partial \rho}{\partial t}\mathrm{dv} \quad \text{but} \quad I = \int_S \underline{\mathbf{J}} \bullet \mathrm{d}\underline{\mathbf{S}}$$

[Think about the sign; charge decreasing implies current flowing out of surface and note the surface is closed]

Gauss' Theorem states $\int_v \nabla \bullet \underline{\mathbf{J}}\,\mathrm{dv} = \oint_S \underline{\mathbf{J}} \bullet \mathrm{d}\underline{\mathbf{S}}$

So that $\int_v \nabla \bullet \underline{\mathbf{J}}\,\mathrm{dv} = -\int_v \frac{\partial \rho}{\partial t}\mathrm{dv}$ or $\nabla \bullet \underline{\mathbf{J}} = -\frac{\partial \rho}{\partial t}$

DISPLACEMENT CURRENT

In magnetostatics we found that $\oint_l \underline{\mathbf{H}} \bullet d\underline{\mathbf{l}} = I$ and hence $\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}}$

But $\nabla \bullet \nabla \times \underline{\mathbf{H}} = 0$ always (!) and $\nabla \bullet \underline{\mathbf{J}} \neq 0$ always!

$\nabla \bullet \underline{\mathbf{J}} = 0$ only when $\dfrac{\partial \rho}{\partial t} = 0$ i.e. STATICS

RESOLUTION OF THE PROBLEM

$$\nabla \bullet \underline{\mathbf{D}} = \rho \qquad \Rightarrow \qquad \nabla \bullet \frac{\partial \underline{\mathbf{D}}}{\partial t} = \frac{\partial \rho}{\partial t}$$

As $\nabla \bullet \underline{\mathbf{J}} = -\dfrac{\partial \rho}{\partial t} \qquad \Rightarrow \qquad \nabla \bullet \underline{\mathbf{J}} = -\nabla \bullet \dfrac{\partial \underline{\mathbf{D}}}{\partial t}$ or $\nabla \bullet \left(\underline{\mathbf{J}} + \dfrac{\partial \underline{\mathbf{D}}}{\partial t} \right) = 0$

Now we can see how we may amend Amperes Law $\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + \dfrac{\partial \underline{\mathbf{D}}}{\partial t}$

$\dfrac{\partial \underline{\mathbf{D}}}{\partial t} =$ Displacement current density [Am^{-2}]

Total effective current $= \underline{\mathbf{J}} + \dfrac{\partial \underline{\mathbf{D}}}{\partial t}$ [Am^{-2}]

$\underline{\mathbf{J}} =$ Conduction current density [Am^{-2}]

$I = \int_S \underline{\mathbf{J}} \bullet d\underline{\mathbf{S}}$ Conduction Current

$I = \int_S \dfrac{\partial \underline{\mathbf{D}}}{\partial t} \bullet d\underline{\mathbf{S}}$ Displacement Current (not a real current)

AMPERE-MAXWELL LAW IN A DIELECTRIC WITH A FINTE CONDUCTIVITY

$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + \frac{\partial \underline{\mathbf{D}}}{\partial t}$ $\quad\quad$ $\underline{\mathbf{J}} = \sigma_C \underline{\mathbf{E}}$ $\quad\quad$ $\underline{\mathbf{D}} = \varepsilon_0 \underline{\mathbf{E}} + \underline{\mathbf{P}}$

$$\nabla \times \underline{\mathbf{H}} = \sigma_C \underline{\mathbf{E}} + \frac{\partial \underline{\mathbf{P}}}{\partial t} + \varepsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

Conduction current
(Motion of free charges
through the medium)

Not related to a motion of
any sort of charge

Motion of the bound polarisation charges
in the vicinity of its nucleus.

In fact we have found that for time varying fields in vacuum ($\sigma_C = 0$, $\underline{\mathbf{P}} = 0$)

$$\nabla \times \underline{\mathbf{H}} = \varepsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

We see a fundamental difference between dynamic and static electrical and magnetic fields.

STATICS:

$\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ are completely independent of each other.

DYNAMICS (examples in vacuum):

When $\frac{\partial \underline{\mathbf{E}}}{\partial t}$ is finite must also have a $\underline{\mathbf{H}}$ field where $\quad \nabla \times \underline{\mathbf{E}} = -\mu_0 \frac{\partial \underline{\mathbf{H}}}{\partial t}$

or when $\frac{\partial \underline{\mathbf{H}}}{\partial t}$ is finite must also have a $\underline{\mathbf{E}}$ field where $\quad \nabla \times \underline{\mathbf{H}} = \varepsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$

In dynamics $\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ are coupled (cannot have one without the other).

MAXWELLS EQUATIONS

Gauss' Law in electricity and magnetism

$$\nabla \bullet \underline{\mathbf{D}} = \rho \qquad [M1]$$

$$\nabla \bullet \underline{\mathbf{B}} = 0 \qquad [M2]$$

Ampere-Maxwell Law

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + \frac{\partial \underline{\mathbf{D}}}{\partial t} \qquad [M3]$$

Faraday Law

$$\nabla \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad [M4]$$

LINEAR AND ISOTROPIC MEDIA	LINEAR, ISOTROPIC AND HOMOGENEOUS MEDIA
$\underline{\mathbf{D}} = \varepsilon_0 \varepsilon_r \underline{\mathbf{E}} \quad \underline{\mathbf{B}} = \mu_0 \mu_r \underline{\mathbf{H}} \quad \underline{\mathbf{J}} = \sigma_C \underline{\mathbf{E}}$	ε_r and μ_r independent of position
$\nabla \bullet \varepsilon_r \underline{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$	$\nabla \bullet \underline{\mathbf{E}} = \frac{\rho}{\varepsilon_r \varepsilon_0}$
$\nabla \bullet \mu_r \underline{\mathbf{H}} = 0$	$\nabla \bullet \underline{\mathbf{H}} = 0$
$\nabla \times \underline{\mathbf{H}} = \sigma_C \underline{\mathbf{E}} + \varepsilon_0 \varepsilon_r \frac{\partial \underline{\mathbf{E}}}{\partial t}$	$\nabla \times \underline{\mathbf{H}} = \sigma_C \underline{\mathbf{E}} + \varepsilon_0 \varepsilon_r \frac{\partial \underline{\mathbf{E}}}{\partial t}$
$\nabla \times \underline{\mathbf{E}} = -\mu_0 \mu_r \frac{\partial \underline{\mathbf{H}}}{\partial t}$	$\nabla \times \underline{\mathbf{E}} = -\mu_0 \mu_r \frac{\partial \underline{\mathbf{H}}}{\partial t}$

GENERAL WAVE EQUATION

Consider a medium in which $\rho = 0$, and that is LINEAR, ISOTROPIC and HOMOGENEOUS (ε_r and μ_r constants, independent of position)

$\underline{\mathbf{D}} = \varepsilon \underline{\mathbf{E}}$ $\quad \varepsilon = \varepsilon_0 \varepsilon_r$ $\quad \underline{\mathbf{B}} = \mu \underline{\mathbf{H}}$ $\quad \mu = \mu_0 \mu_r$ $\quad \underline{\mathbf{J}} = \sigma_C \underline{\mathbf{E}}$

$\nabla \bullet \underline{\mathbf{E}} = 0$ $\qquad \nabla \bullet \underline{\mathbf{H}} = 0$

$\nabla \times \underline{\mathbf{H}} = \sigma_C \underline{\mathbf{E}} + \varepsilon \frac{\partial \underline{\mathbf{E}}}{\partial t}$ $\qquad \nabla \times \underline{\mathbf{E}} = -\mu \frac{\partial \underline{\mathbf{H}}}{\partial t}$

Starting with $\quad \nabla \times \underline{\mathbf{H}} = \sigma_C \underline{\mathbf{E}} + \varepsilon \frac{\partial \underline{\mathbf{E}}}{\partial t}$

Take the curl of both sides

$$\nabla \times \nabla \times \underline{\mathbf{H}} = \sigma_C \nabla \times \underline{\mathbf{E}} + \varepsilon \frac{\partial (\nabla \times \underline{\mathbf{E}})}{\partial t}$$

Using $\nabla \times \nabla \times \underline{\mathbf{F}} = \nabla(\nabla \bullet \underline{\mathbf{F}}) - \nabla^2 \underline{\mathbf{F}}$ and $\nabla \times \underline{\mathbf{E}} = -\mu \frac{\partial \underline{\mathbf{H}}}{\partial t}$

$$\nabla\nabla \bullet \underline{\mathbf{H}} - \nabla^2 \underline{\mathbf{H}} = -\sigma_C \mu \frac{\partial \underline{\mathbf{H}}}{\partial t} - \varepsilon\mu \frac{\partial^2 \underline{\mathbf{H}}}{\partial t^2}$$

Since $\nabla \bullet \underline{\mathbf{H}} = 0$

$$\nabla^2 \underline{\mathbf{H}} = \sigma_C \mu \frac{\partial \underline{\mathbf{H}}}{\partial t} + \varepsilon\mu \frac{\partial^2 \underline{\mathbf{H}}}{\partial t^2}$$

Starting with $\quad \nabla \times \underline{\mathbf{E}} = -\mu \frac{\partial \underline{\mathbf{H}}}{\partial t}$

Take the curl of both sides

$$\nabla \times \nabla \times \underline{\mathbf{E}} = -\mu \frac{\partial (\nabla \times \underline{\mathbf{H}})}{\partial t}$$

Using $\nabla \times \nabla \times \underline{\mathbf{F}} = \nabla(\nabla \bullet \underline{\mathbf{F}}) - \nabla^2 \underline{\mathbf{F}}$ and $\nabla \times \underline{\mathbf{H}} = \sigma_C \underline{\mathbf{E}} + \varepsilon \frac{\partial \underline{\mathbf{E}}}{\partial t}$

$$\nabla(\nabla \bullet \underline{\mathbf{E}}) - \nabla^2 \underline{\mathbf{E}} = -\mu\sigma_C \frac{\partial \underline{\mathbf{E}}}{\partial t} - \mu\varepsilon \frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2}$$

Since $\nabla \bullet \underline{\mathbf{E}} = 0$

$$\nabla^2 \underline{\mathbf{E}} = \mu\sigma_C \frac{\partial \underline{\mathbf{E}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2}$$

We have found the general wave equation

$$\nabla^2 \underline{\mathbf{F}} = \mu\sigma_C \frac{\partial \underline{\mathbf{F}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \underline{\mathbf{F}}}{\partial t^2} \qquad \text{“The Equation of Telegraphy”}$$

where $\underline{\mathbf{F}}$ could be $\underline{\mathbf{D}}, \underline{\mathbf{E}}, \underline{\mathbf{B}}, \underline{\mathbf{H}}$ or $\underline{\mathbf{J}}$ or even currents propagating along cables.

THE WAVE EQUATION AND THE DIFFUSION EQUATION

CASE 1
In an ideal Dielectric medium ($\sigma_C = 0$) or in a vacuum $\sigma_C = 0$, $\mu_r = \varepsilon_r = 1$

$$\nabla^2 \underline{\mathbf{F}} = \mu\varepsilon \frac{\partial^2 \underline{\mathbf{F}}}{\partial t^2} \qquad \Rightarrow \qquad \nabla^2 \underline{\mathbf{F}} = \mu_0\varepsilon_0 \frac{\partial^2 \underline{\mathbf{F}}}{\partial t^2}$$

CASE 2
Alternatively in a medium of high conductivity we find

$$\left|\mu\sigma_C \frac{\partial \underline{\mathbf{F}}}{\partial t}\right| >> \left|\mu\varepsilon \frac{\partial^2 \underline{\mathbf{F}}}{\partial t^2}\right| \qquad \Rightarrow \sigma_C \left|\frac{\partial \underline{\mathbf{F}}}{\partial t}\right| >> \varepsilon \left|\frac{\partial^2 \underline{\mathbf{F}}}{\partial t^2}\right|$$

Then we get the “Diffusion Equation” $\nabla^2 \underline{\mathbf{F}} = \mu\sigma_C \dfrac{\partial \underline{\mathbf{F}}}{\partial t}$

WAVE EQUATION IN FREE SPACE – Plane Wave soln. of $\nabla^2 \underline{\mathbf{F}} = \mu_0\varepsilon_0 \dfrac{\partial^2 \underline{\mathbf{F}}}{\partial t^2}$

PLANE WAVE $\Rightarrow$ There exists a plane on which the field components do not vary spatially, i.e. the magnitude of the field vectors vary with time but are independent of position on the plane. The plane is called the “Plane of Polarisation” and is also often called the “Wavefront”.

e.g. $\underline{\mathbf{F}}(x, y, z) = \underline{\mathbf{j}}F(x)\sin(x - vt)$ is a plane wave travelling in the positive x-direction with a velocity v . The wave is polarised in the yz-plane, in this case along the y-axis. In the yz-plane the field components do not vary spatially i.e. application of the operators $\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial z}$ to this $\underline{\mathbf{F}}$ gives a result of zero.

The velocity of propagation in free space is $v = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$ [ms^{-1}]

We have seen that $\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ are "coupled" by the Maxwell curl equations, and since $\underline{\mathbf{F}}$ represents any of the field components these waves are called "ELECTROMANETIC WAVES". It was Maxwell who observed "that the velocity of electromagnetic waves was the same as that of light and so light was an electromagnetic wave phenomenon" – the unification of Electricity and Magnetism with optics.

In S.I. units $c = 2.99792458 \times 10^8$ ms^{-1} [DEFINED]

$\mu_0 = 4\pi \times 10^{-7}$ Hm^{-1} [DEFINED]

$c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$ defines $\varepsilon_0 = 8.854187814\ldots \times 10^{-12}$ Fm^{-1}

In a dielectric $v = \dfrac{1}{\sqrt{\mu\varepsilon}} = \dfrac{c}{\sqrt{\mu_r \varepsilon_r}}$

Refractive Index $n = \dfrac{c}{v} = \dfrac{\sqrt{\mu\varepsilon}}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\mu_r \varepsilon_r}$

In diamagnetic / paramagnetic media $\mu_r \approx 1$ and so $n = \sqrt{\varepsilon_r}$ or $n^2 = \varepsilon_r$

n is an optical quantity and ε_r is an electrical quantity. Unfortunately n and ε_r vary with the wavelength (frequency) of the wave – real media are "DISPERSIVE". Very difficult to measure n and ε_r at the same wavelength…

PLANE WAVES IN A LINEAR, ISOTROPIC, AND HOMOGENEOUS MEDIUM WITH $\sigma_C = 0$.

Consider the wave solution for $\underline{\mathbf{E}}$ propagating in the positive *x*-direction that is of the form $\underline{\mathbf{E}}(x-vt) = \underline{\mathbf{i}}\,E_x(x-vt) + \underline{\mathbf{j}}\,E_y(x-vt) + \underline{\mathbf{k}}\,E_z(x-vt)$ (note plane wave so the operators $\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial z}$ give a result of zero).

$$\nabla \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} = -\mu \frac{\partial \underline{\mathbf{H}}}{\partial t}$$

So we can write
$$\nabla \times \underline{\mathbf{E}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\nabla \times \underline{\mathbf{E}} = \underline{\mathbf{i}}\,\{0\} \quad + \underline{\mathbf{j}}\left\{-\frac{\partial E_z}{\partial x}\right\} \quad + \underline{\mathbf{k}}\left\{\frac{\partial E_y}{\partial x}\right\}$$

$$\nabla \times \underline{\mathbf{E}} = -\mu\left[\underline{\mathbf{i}}\,\frac{\partial H_x}{\partial t} \quad + \underline{\mathbf{j}}\,\frac{\partial H_y}{\partial t} \quad + \underline{\mathbf{k}}\,\frac{\partial H_z}{\partial t}\right]$$

$\therefore \qquad H_x = 0$

Similarly $\nabla \times \underline{\mathbf{H}} = \varepsilon \dfrac{\partial \underline{\mathbf{E}}}{\partial t}$ $\qquad$ [$\sigma_C = 0$, and $\underline{\mathbf{D}} = \varepsilon \underline{\mathbf{E}}$]

$$\nabla \times \underline{\mathbf{H}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{i}}\,\{0\} \quad + \underline{\mathbf{j}}\left\{-\frac{\partial H_z}{\partial x}\right\} \quad + \underline{\mathbf{k}}\left\{\frac{\partial H_y}{\partial x}\right\}$$

$$\nabla \times \underline{\mathbf{H}} = -\varepsilon\left[\underline{\mathbf{i}}\,\frac{\partial E_x}{\partial t} \quad + \underline{\mathbf{j}}\,\frac{\partial E_y}{\partial t} \quad + \underline{\mathbf{k}}\,\frac{\partial E_z}{\partial t}\right]$$

$\therefore \qquad E_x = 0$

$\therefore$ In a linear, isotropic medium where ε_r and μ_r are scalar constants so that be "$\underline{\mathbf{D}}$ and $\underline{\mathbf{E}}$" and "$\underline{\mathbf{B}}$ and $\underline{\mathbf{H}}$" are parallel. No component of the wave field is in the x-direction. All wave components lie in the plane of the wavefront, transverse (perpendicular) to the direction of propagation.
Plane electromagnetic waves are a TRANSVERSE wave motion in an "isotropic" medium – called TEM mode (no longitudinal component of the electromagnetic field.
From y-components (top line) and z-components (second line)

$$\frac{\partial E_z}{\partial x} = \mu \frac{\partial H_y}{\partial t} \qquad \text{and} \qquad -\frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t} \qquad \text{and} \qquad \frac{\partial H_y}{\partial x} = \varepsilon \frac{\partial E_z}{\partial t}$$

Note that only $\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ components at right angles to each other are related by Maxwells equations – suggests that $\underline{\mathbf{E}}$ is perpendicular (orthogonal) to $\underline{\mathbf{H}}$.

EXAMPLE
Possible solution for $\underline{\mathbf{E}} = \underline{\mathbf{k}}\, E_0 \cos(\omega t - kx)$, if so what is the solution for $\underline{\mathbf{H}}$?

$$\nabla \times \underline{\mathbf{E}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_0 \cos(\omega t - kx) \end{vmatrix} = -\underline{\mathbf{j}} E_0 k \sin(\omega t - kx)$$

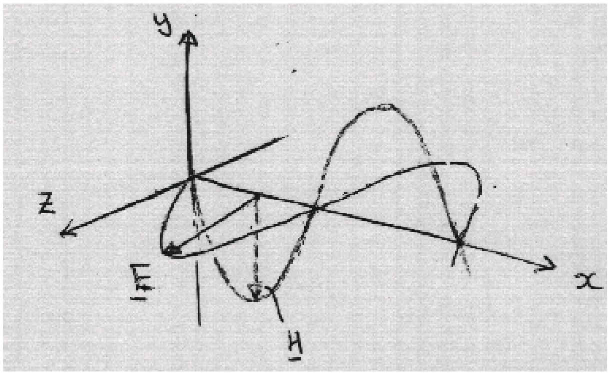
$$\nabla \times \underline{\mathbf{E}} = -\mu \left[\underline{\mathbf{i}} \frac{\partial H_x}{\partial t} + \underline{\mathbf{j}} \frac{\partial H_y}{\partial t} + \underline{\mathbf{k}} \frac{\partial H_z}{\partial t} \right]$$

$$\therefore -E_0 k \sin(\omega t - kx) = -\mu \frac{\partial H_y}{\partial t}.$$

Integrate: $H_y = -\dfrac{1}{\mu}\dfrac{k}{\omega} E_0 \cos(\omega t - kx)$

$$\therefore \qquad \underline{\mathbf{H}} = -\underline{\mathbf{j}} H_0 \cos(\omega t - kx)$$

$$H_0 = \frac{k}{\omega \mu} E_0$$



$\underline{\mathbf{H}}$ is in ANTIPHASE with $\underline{\mathbf{E}}$. Transverse plane polarised wave $\underline{\mathbf{H}}$ and $\underline{\mathbf{E}}$ perpendicular to each other.

THE COMPLEX REPRESENTATION OF ELECTROMAGNETIC WAVES

We discovered in the last lecture that we had to solve equations of the form

$$\nabla^2 \underline{\mathbf{E}} = \mu\sigma_C \frac{\partial \underline{\mathbf{E}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2}$$

The solution to this equation can be written in the complex form

$$\underline{\mathbf{E}}(x, y, z, t) = \underline{\tilde{\mathbf{E}}}(x, y, z)\exp(j\omega t)$$

In general $\underline{\tilde{\mathbf{E}}}(x, y, z)$ is a complex number (vector) that varies spatially but is independent of time.

$\omega = 2\pi f =$ Angular wave Frequency [Rad s^{-1}], and $f =$ Wave Frequency [Hz]

Note $\frac{\partial \underline{\mathbf{E}}}{\partial t} = j\omega \underline{\mathbf{E}}$ and $\frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2} = -\omega^2 \underline{\mathbf{E}}$

Remember physical "wave fields" are REAL functions of position and time. When solving a problem we must recover the "real part" from the solution – note this is not as obvious as it may seem because $\underline{\tilde{\mathbf{E}}}(x, y, z)$ can be a complex number. Using the complex notation the wave equation becomes

$$\nabla^2 \underline{\mathbf{E}} = j\omega\mu\sigma_C \underline{\mathbf{E}} - \mu\varepsilon\omega^2 \underline{\mathbf{E}}$$

Remember we could replace $\underline{\mathbf{E}}$ with $\underline{\mathbf{D}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{H}}$ or $\underline{\mathbf{J}}$ and the equation would still be valid.

THE SINGLE PROGRESSIVE (COMPLEX) PLANE WAVE IN AN IDEAL DIELECTRIC ($\sigma_C = 0$).

Must find a solution for $\underline{\mathbf{E}}$ that satisfies the equation $\nabla^2 \underline{\mathbf{E}} = -\omega^2 \mu\varepsilon \underline{\mathbf{E}}$

Assume the solution is of the form $\underline{\mathbf{E}}(x, y, z, t) = \underline{\tilde{\mathbf{E}}}_0 \exp[j(\omega t - kx)]$ COHERNET TIME HARMONIC WAVE

Amplitude of the wave oscillation (Complex Constant)

k is called the spatial frequency or wavenumber.

jk is called the propagation constant.

$\phi = -kx$ is the phase of the wave (so in any plane $x =$ constant is a plane of constant phase).

$$\nabla^2 \underline{\mathbf{E}} = \frac{\partial^2 \underline{\mathbf{E}}}{\partial x^2} = -k^2 \underline{\mathbf{E}} \qquad \text{and} \qquad \frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2} = -\omega^2 \underline{\mathbf{E}}$$

$$\therefore -k^2 \underline{\mathbf{E}} = -\mu\varepsilon\omega^2 \underline{\mathbf{E}} \text{ and hence } k = \pm\omega\sqrt{\mu\varepsilon} = \pm\frac{\omega}{v} = \pm\frac{2\pi}{\lambda} = \pm\frac{\omega n}{c}$$

- $k =$ positive root $\Rightarrow$ Wave propagating in the positive *x*-direction.
- $k = \pm\omega\sqrt{\mu\varepsilon}$ which is a real number – peak amplitude does not change as wave propagates in the *x*-direction. Wave is said to be “un-attenuated”.

THE PHYSICAL SOLUTION – we want the real part of $\underline{\tilde{\mathbf{E}}}(x, y, z)$ subject to the appropriate boundary conditions.

We have $\underline{\mathbf{E}}(x, y, z, t) = \underline{\tilde{\mathbf{E}}}_0 \exp[j(\omega t - kx)]$ but we know that there is no wave component of the electric field in the *x*-direction.

$$\therefore \underline{\mathbf{E}}(x, y, z, t) = \left(\underline{\mathbf{j}}\tilde{E}_{y0} + \underline{\mathbf{k}}\tilde{E}_{z0} \right) \exp[j(\omega t - kx)]$$

<u>Example 1</u>

If $\tilde{E}_{y0} = E_{y0}$ and $\tilde{E}_{z0} = E_{z0}$ (both E_{y0} and E_{z0} are real constants)

Then $\underline{\mathbf{E}}(x, y, z, t) = \left(\underline{\mathbf{j}}E_{y0} + \underline{\mathbf{k}}E_{z0} \right) \cos(\omega t - kx)$

<u>Example 2</u>

If $\tilde{E}_{y0} = -jE_{y0}$ and $\tilde{E}_{z0} = -jE_{z0}$ (both E_{y0} and E_{z0} are real constants)

Then $\underline{\mathbf{E}}(x, y, z, t) = \left(\underline{\mathbf{j}}E_{y0} + \underline{\mathbf{k}}E_{z0} \right) \sin(\omega t - kx)$

<u>Example 3</u>

If $\tilde{E}_{y0} = -jE_{y0}$ and $\tilde{E}_{z0} = -je^{j\delta}E_{z0}$ (E_{y0}, E_{z0} and δ are real constants)

Then $\underline{\mathbf{E}}(x, y, z, t) = \underline{\mathbf{j}}E_{y0} \sin(\omega t - kx) + \underline{\mathbf{k}}E_{z0} \sin(\omega t - kx + \delta)$

The *z*-component E_z leads the *y*-component E_y by the phase angle δ.

Now we have the electric field component, how do we get the magnetic field component?

$$\nabla \times \underline{\underline{\mathbf{H}}} = \varepsilon \frac{\partial \underline{\underline{\mathbf{E}}}}{\partial t} \qquad\qquad \nabla \times \underline{\underline{\mathbf{E}}} = -\mu \frac{\partial \underline{\underline{\mathbf{H}}}}{\partial t}$$

$$\nabla \times \underline{\underline{\mathbf{E}}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \tilde{E}_y & \tilde{E}_z \end{vmatrix}$$

$$\nabla \times \underline{\underline{\mathbf{E}}} = \underline{\mathbf{i}}\,\{0\} \quad + \underline{\mathbf{j}}\left\{-\frac{\partial}{\partial x}\left(\tilde{E}_{z0}\exp\left[j(\omega t - kx)\right]\right)\right\} \quad + \underline{\mathbf{k}}\left\{\frac{\partial}{\partial x}\left(\tilde{E}_{y0}\exp\left[j(\omega t - kx)\right]\right)\right\}$$

$$\frac{\partial \underline{\underline{\mathbf{H}}}}{\partial t} = \underline{\mathbf{i}}\,\frac{\partial}{\partial t}\left(\tilde{H}_{x0}\exp\left[j(\omega t - kx)\right]\right) \quad + \underline{\mathbf{j}}\,\frac{\partial}{\partial t}\left(\tilde{H}_{y0}\exp\left[j(\omega t - kx)\right]\right) \quad + \underline{\mathbf{k}}\,\frac{\partial}{\partial t}\left(\tilde{H}_{z0}\exp\left[j(\omega t - kx)\right]\right)$$

$$\nabla \times \underline{\underline{\mathbf{E}}} = -\mu \frac{\partial \underline{\underline{\mathbf{H}}}}{\partial t}$$

$$\Rightarrow \qquad \tilde{H}_{x0} = 0$$

$$\Rightarrow \qquad jk\tilde{E}_{z0} = -\mu j\omega \tilde{H}_{y0} \quad \text{and hence} \quad \tilde{H}_{y0} = -\frac{k}{\mu\omega}\tilde{E}_{z0}$$

$$\Rightarrow \qquad -jk\tilde{E}_{y0} = -\mu j\omega \tilde{H}_{z0} \qquad \text{and hence} \quad \tilde{H}_{z0} = \frac{k}{\mu\omega}\tilde{E}_{y0}$$

$$\underline{\underline{\mathbf{H}}}(x,y,z,t) = \left(-\underline{\mathbf{j}}\left(\frac{k}{\mu\omega}\tilde{E}_{z0}\right) + \underline{\mathbf{k}}\left(\frac{k}{\mu\omega}\tilde{E}_{y0}\right)\right)\exp\left[j(\omega t - kx)\right]$$

$$\underline{\underline{\mathbf{H}}}(x,y,z,t) = \left(\underline{\mathbf{j}}\tilde{H}_{y0} + \underline{\mathbf{k}}\tilde{H}_{z0}\right)\exp\left[j(\omega t - kx)\right]$$

THE POLARISATION STATE OF AN ELECTROMAGNETIC WAVE

Consider the wave with $\underline{\mathbf{E}} = \underline{\mathbf{j}}\, E_{y0} \sin(\omega t - kx) + \underline{\mathbf{k}}\, E_{z0} \sin(\omega t - kx + \delta)$

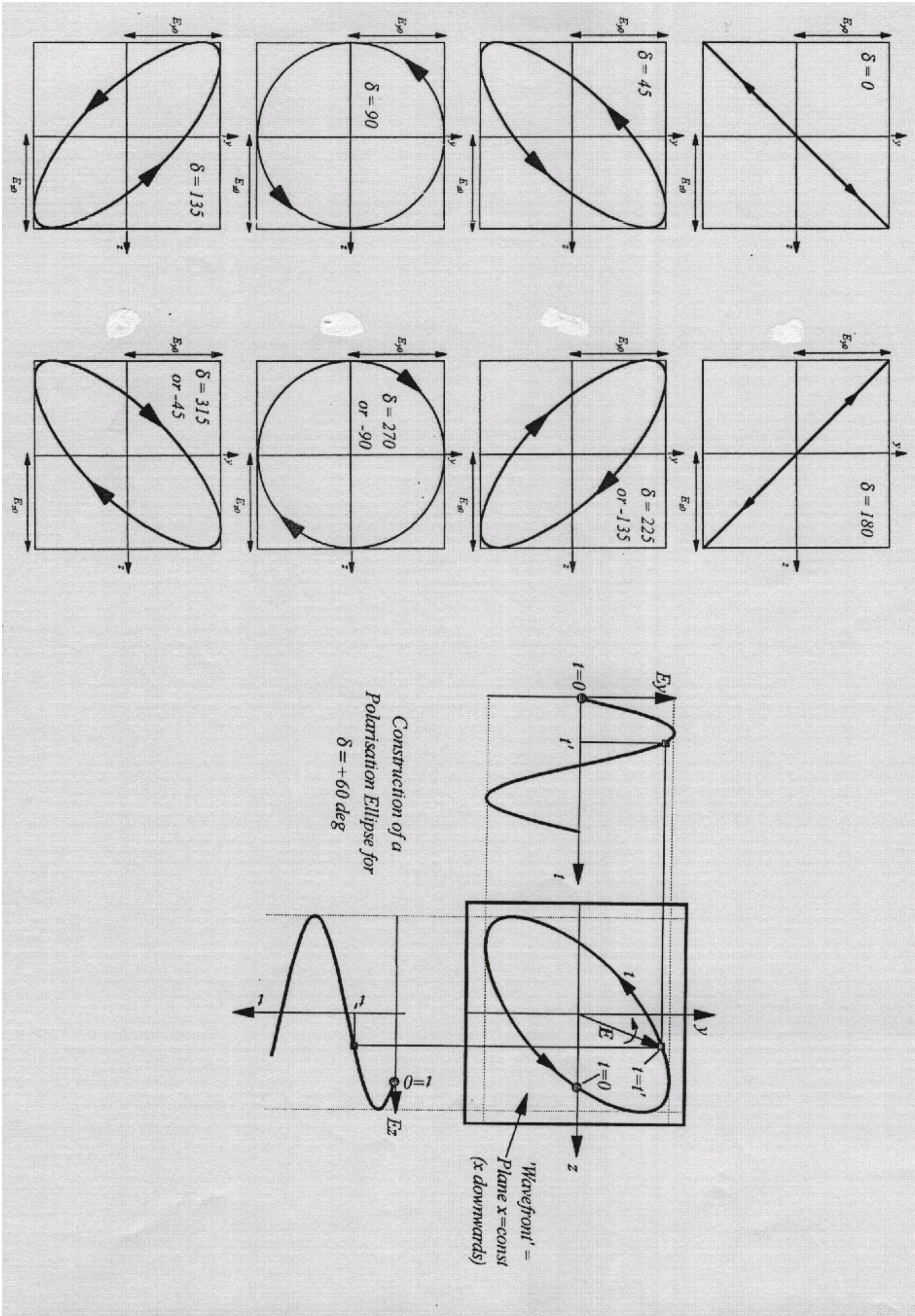
i.e. $E_y = E_{y0} \sin(\omega t - kx)$ and $E_z = E_{z0} \sin(\omega t - kx + \delta)$

Various polarisation sates of the wave are possible depending on the relative magnitudes and phases of the two "*E*" components.

Consider any plane *x*=constant. Any such plane of constant phase and is called a "plane of polarisation" or a "wavefront". What happens in this plane as time varies?

Consider the plane *x*=0. $E_y = E_{y0} \sin(\omega t)$ and $E_z = E_{z0} \sin(\omega t + \delta)$

- If $\delta = 0$ or $\delta = \pi$, the polarisation is LINEAR. The amplitude of the total electric vector varies between zero and $E = \sqrt{E_{y0}^2 + E_{z0}^2}$.
- If $\delta = \pm \frac{\pi}{2}$ and $E_{y0} = E_{z0}$ the polarisation is CIRCULAR and the magnitude of the total electric vector is independent of time.
- If $0 < \delta < \pi$ figure s described in an "anticlockwise" sense.
- If $-\pi < \delta < 0$ figure s described in a "clockwise" sense.
- Otherwise the polarisation of the wave is elliptical. The magnitude of the total electric vector is never zero.


δ = 0
δ = 180
δ = 45
δ = 225
or -135
δ = 90
δ = 270
or -90
δ = 135
δ = 315
or -45
Ey
t=0
t'
t
E
t=t'
z
y
Ez
'Wavefront' =
Plane x=const
(x downwards)
Construction of a
Polarisation Ellipse for
δ=+60 deg

THE MAGNEIC FIELD

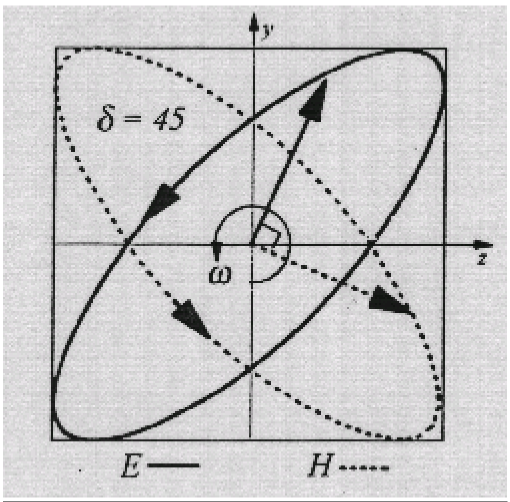
$$E_y = E_{y0}\sin(\omega t - kx) \qquad\qquad H_z = \frac{k}{\mu\omega}E_{y0}\sin(\omega t - kx)$$

$$\nabla \times \underline{\mathbf{E}} = -\mu \frac{\partial \underline{\mathbf{H}}}{\partial t}$$

$$E_z = E_{z0}\sin(\omega t - kx + \delta) \qquad\qquad H_y = -\frac{k}{\mu\omega}E_{z0}\sin(\omega t - kx + \delta)$$

$$H_y = \frac{k}{\mu\omega}E_{z0}\sin(\omega t - kx + \delta \pm \pi)$$

We see (again!) that $H_y = -\frac{k}{\mu\omega}E_z$ and $H_z = \frac{k}{\mu\omega}E_y$



ORTHOGONALITY $\underline{\mathbf{E}} \bullet \underline{\mathbf{H}} = E_x H_x + E_y H_y + E_z H_z = 0$

$\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ are perpendicular to each other at all times!

WAVE IMPEDANCE/ INTRINSIC IMPEDANCE OF A MEDIUM

$E_y = E_{y0}\sin(\omega t - kx)$ and $E_z = E_{z0}\sin(\omega t - kx + \delta)$

The magnitude of the electric field is $E = \sqrt{E_y^2 + E_z^2}$

Similarly $H = \sqrt{H_y^2 + H_z^2} = \dfrac{k}{\mu\omega}\sqrt{E_z^2 + E_y^2} = \dfrac{\sqrt{\varepsilon\mu}\omega}{\mu\omega}E = \sqrt{\dfrac{\varepsilon}{\mu}}E$

$\therefore \dfrac{E}{H} = \sqrt{\dfrac{\mu}{\varepsilon}}$ [Ohm] INTRISIC IMPEDANCE OF THE MEDIUM

In free space $\dfrac{E}{H} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377\Omega$. In an ideal dielectric $\dfrac{E}{H} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\sqrt{\dfrac{\mu_r}{\varepsilon_r}}$ (real quantity purely resistive)

[Note: $\dfrac{E}{H} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\sqrt{\dfrac{\mu_r}{\varepsilon_r}}$ and $\underline{\mathbf{D}} = \varepsilon\underline{\mathbf{E}}$ so $|\underline{\mathbf{D}}| = \sqrt{\mu\varepsilon}|\underline{\mathbf{H}}|$ etc.]

ENERGY TRANSPORTED IN EM-WAVE

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} \quad \Rightarrow \quad \varepsilon_0\varepsilon_r E^2 = \mu_0\mu_r H^2$$

$\dfrac{\varepsilon_0\varepsilon_r E^2}{2}$ = ENERGY DENSITY OF ELECTRIC FIELD [Jm^{-3}]

$\dfrac{\varepsilon_0\varepsilon_r H^2}{2}$ = ENERGY DENSITY OF MAGNETIC FIELD [Jm^{-3}]

$\varepsilon_0\varepsilon_r E^2 = \mu_0\mu_r H^2$ $\Rightarrow$Wave energy is equally divided between electric and magnetic components of field in dielectric medium.

ELECTROMAGNETIC WAVES IN MEDIA OF FINITE CONDUCTIVITY.

Relaxation Time of the Medium

Linear, homogeneous, isotropic medium of conductivity σ_c which contains free charge of volume density ρ

Equation of continuity $\nabla \bullet \underline{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0$

$$\underline{\mathbf{J}} = \sigma_C \underline{\mathbf{E}} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla \bullet \sigma_C \underline{\mathbf{E}}$$

$$\underline{\mathbf{D}} = \varepsilon \underline{\mathbf{E}} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = -\frac{\sigma_C}{\varepsilon} \nabla \bullet \underline{\mathbf{D}}$$

$$\nabla \bullet \underline{\mathbf{D}} = \rho \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = -\frac{\sigma_C}{\varepsilon} \rho \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\sigma_C}{\varepsilon} \rho = 0$$

Solution of which is $\rho(x, y, z, t) = \rho(x, y, z) \exp\left(-\frac{t}{\tau}\right)$ where $\tau = \frac{\varepsilon}{\sigma_C}$ and is called the "relaxation time".

e.g. For copper $\sigma_c = 5.8 \times 10^7 \quad \Omega^{-1} m^{-1}$ and $\varepsilon_r \approx 1$ so that $\tau \approx 10^{-18}$ s
For pure water $\sigma_c \approx 10^{-5} \quad \Omega^{-1} m^{-1}$ and $\varepsilon_r \approx 80$ so that $\tau \approx 10^{-6}$ s

$\therefore$ If a free charge density is present in a conducting medium, it decays away at a rate that is independent of any applied fields. $\Rightarrow$ Eventually all the free charge resides on the surface of the medium – A well know result in Electrostatics of Conductors!

It is impossible to create a stable free charge distribution in a conducting medium.

Plane Waves in a Conducting Medium

Assume plane wave travelling in the *x*-direction $\underline{\mathbf{E}} = \underline{\tilde{\mathbf{E}}}_0 \exp[j(\omega t - kx)]$

Wave Equation: $\nabla^2 \underline{\mathbf{E}} = \mu\sigma_C \frac{\partial \underline{\mathbf{E}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2}$

$\therefore \qquad -k^2 \underline{\mathbf{E}} = j\omega\mu\sigma_C \underline{\mathbf{E}} - \omega^2 \mu\varepsilon \underline{\mathbf{E}}$

or $\qquad k^2 = \omega^2 \mu\varepsilon - j\omega\mu\sigma_C \qquad (1)$

If we let $k = \alpha - j\beta$ then $k^2 = \alpha^2 - \beta^2 - 2j\alpha\beta \qquad (2)$

So using (1) and (2) we get

$$\alpha^2 - \beta^2 = \omega^2 \mu\varepsilon$$

and

$$2\alpha\beta = \omega\mu\sigma_C$$

Which we need to solve for α and β

CASE A: "Poor Conductor" $\quad \Rightarrow \quad \left|\mu\sigma_C \frac{\partial \underline{\mathbf{E}}}{\partial t}\right| << \left|\mu\varepsilon \frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2}\right|$

$$\Rightarrow \quad 2\alpha\beta << \alpha^2 - \beta^2$$

For $\sigma_c = 0$ we see that $\beta = 0$ and $k^2 = \alpha^2 = \omega^2 \mu\varepsilon$

Alternative approach for $\sigma_c = 0$ wave equation becomes $\nabla^2 \underline{\mathbf{E}} = \mu\varepsilon \frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2}$

With $\underline{\mathbf{E}} = \underline{\tilde{\mathbf{E}}}_0 \exp[j(\omega t - kx)]$ we get $\qquad -k^2 \underline{\mathbf{E}} = -\omega^2 \mu\varepsilon \underline{\mathbf{E}}$

$$k^2 = \omega^2 \mu\varepsilon$$

$$k = \omega\sqrt{\mu\varepsilon}$$

Since $k = \frac{2\pi}{\lambda} = \frac{\omega}{v} = \frac{\omega n}{c}$ we see that when $\sigma_c = 0$ both k and n are real.

CASE B: "Good Conductor" $\Rightarrow \quad \left|\mu\sigma_C \frac{\partial \underline{\mathbf{E}}}{\partial t}\right| >> \left|\mu\varepsilon \frac{\partial^2 \underline{\mathbf{E}}}{\partial t^2}\right|$

$$\Rightarrow \quad \omega\mu\sigma_C >> \omega^2\mu\varepsilon$$

$$\sigma_C >> \omega\varepsilon$$

$$\omega << \frac{\sigma_C}{\varepsilon}$$

$\omega\mu\sigma_C >> \omega^2\mu\varepsilon \;\Rightarrow \qquad 2\alpha\beta >> \alpha^2 - \beta^2$

So if we assume $\alpha^2 - \beta^2 \approx 0$ and hence $\alpha^2 \approx \beta^2 \approx \frac{\omega\mu\sigma_C}{2}$

Alternate approach: For a good conductor $\nabla^2 \underline{\mathbf{E}} \approx \mu\sigma_C \frac{\partial \underline{\mathbf{E}}}{\partial t}$

$-k^2\underline{\mathbf{E}} = j\omega\mu\sigma_C\underline{\mathbf{E}}$ and $k^2 = -j\omega\mu\sigma_C$. So with $k = \alpha - j\beta$

$$k^2 = \alpha^2 - \beta^2 - 2j\alpha\beta = -j\omega\mu\sigma_C$$

and $\alpha = \beta$ and $\alpha^2 = \beta^2 = \frac{\omega\mu\sigma_C}{2}$

Remember $\underline{\mathbf{E}} = \underline{\tilde{\mathbf{E}}}_0 \exp\left[j(\omega t - kx)\right]$

We know from before that since the *x*-component of the Maxwell curl equations are zero there is no wave component of $\underline{\mathbf{E}}$ or $\underline{\mathbf{H}}$ in the *x*-direction.

For simplicity we assume $\underline{\mathbf{E}} = \underline{\mathbf{j}}\tilde{E}_{y0} \exp[j(\omega t - kx)]$

Inserting $k = \alpha - j\beta$ gives

$$\underline{\mathbf{E}} = \underline{\mathbf{j}}\tilde{E}_{y0} \exp[-\beta x]\exp[j(\omega t - \alpha x)]$$

$\therefore$ The electric field is attenuated as the wave propagates
and $\beta =$ Attenuation constant [m^{-1}]

The magnetic field is obtained from the specified electric field using the Maxwell curl $\underline{\mathbf{E}}$ equation: $\nabla \times \underline{\mathbf{E}} = -\mu \dfrac{\partial \underline{\mathbf{H}}}{\partial t}$

$$\nabla \times \underline{\mathbf{E}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & \tilde{E}_y & 0 \end{vmatrix}$$

$$\nabla \times \underline{\mathbf{E}} = \underline{\mathbf{i}}\,\{0\} \quad + \underline{\mathbf{j}}\,\{0\} \quad + \underline{\mathbf{k}}\left\{\frac{\partial}{\partial x}\left(\tilde{E}_{y0}\exp[j(\omega t - kx)]\right)\right\}$$

We see that $\underline{\mathbf{H}} = \underline{\mathbf{k}}\tilde{H}_{z0}\exp[j(\omega t - kx)]$

$$\frac{\partial \underline{\mathbf{H}}}{\partial t} = \underline{\mathbf{i}}\,(0) \quad + \underline{\mathbf{j}}\,(0) \quad + \underline{\mathbf{k}}\frac{\partial}{\partial t}\left(\tilde{H}_{z0}\exp[j(\omega t - kx)]\right)$$

Using $\nabla \times \underline{\mathbf{E}} = -\mu \dfrac{\partial \underline{\mathbf{H}}}{\partial t}$

$\Rightarrow \qquad -jk\tilde{E}_{y0} = -\mu j\omega \tilde{H}_{z0}$ $\qquad$ and hence $\tilde{H}_{z0} = \dfrac{k}{\mu\omega}\tilde{E}_{y0}$

$$\underline{\mathbf{H}} = \underline{\mathbf{k}}\left(\frac{k}{\mu\omega}\tilde{E}_{y0}\right)\exp[j(\omega t - kx)]$$

Inserting $k = \alpha - j\beta$ gives

$$\underline{\mathbf{H}} = \underline{\mathbf{k}}\frac{(\alpha - j\beta)}{\mu\omega}\tilde{E}_{y0}\exp[-\beta x] \;\; \exp[\,j(\omega t - \alpha x)]$$

Writing $\alpha - j\beta = \sqrt{\alpha^2 + \beta^2}\exp(-j\phi)$ where $\tan\phi = \dfrac{\beta}{\alpha}$

$$\underline{\mathbf{H}} = \underline{\mathbf{k}}\frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega}\,\tilde{E}_{y0}\exp[-\beta x]\exp[j(\omega t - \alpha x - \phi)]$$

But $\alpha^2 = \beta^2 = \dfrac{\omega\mu\sigma_C}{2}$

$$\underline{\mathbf{H}} = \underline{\mathbf{k}}\sqrt{\frac{\sigma_c}{\mu\omega}}\,\tilde{E}_{y0}\exp[-\beta x]\exp[j(\omega t - \alpha x - \phi)]$$

$$\underline{\mathbf{E}} = \underline{\mathbf{j}}\tilde{E}_{y0}\exp[-\beta x]\exp[j(\omega t - \alpha x)]$$

- Like the electric field, the magnetic field is attenuated as the wave propagates.
- We note also that the magnetic field lags behind the electric field by a phase angle ϕ. In a good conductor $\alpha = \beta$ so that $\phi = \frac{\pi}{4}$.
- In a good conductor we define the "skin depth" $\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\mu\sigma_C}}$ [m].

  When the wave impinges on a good conductor practically all the transmitted energy is absorbed in a few "skin depths" – i.e. converted to "Joule Heat" within the material.

WHAT IS A GOOD CONDUCTOR?

If $\omega << \frac{\sigma_C}{\varepsilon}$ then the material is a good conductor!

For example:
Pure water: EM radiation at 5×10^{14} Hz, $\varepsilon_r = 2.33$ and $\sigma_C = 10^{-5}\,\Omega^{-1}m^{-1}$

$$\frac{\sigma_C}{\varepsilon_0\varepsilon_r} = \frac{10^{-5}}{8.85\times10^{-12}\times2.33} = 4.85\times10^5 \text{ which is } << 2\pi\times5\times10^{14}$$

Hence at these frequencies, fresh water is a very poor conductor! Waves transmitted without much loss.

Sea water: EM radiation at 1000 Hz, $\varepsilon_r = 80$ and $\sigma_C = 5\Omega^{-1}m^{-1}$

$$\frac{\sigma_C}{\varepsilon_0\varepsilon_r} = \frac{5}{8.85\times10^{-12}\times80} = 7\times10^9 \text{ which is } >> 2\pi\times10^3$$

Hence at these frequencies, seawater is a very good conductor! Waves rapidly attenuated.

Plane Waves in a Conducting Medium (General Solution)

Assume plane wave travelling in the *x*-direction $\underline{\mathbf{E}} = \underline{\tilde{\mathbf{E}}}_0 \exp\left[j(\omega t - kx)\right]$

Wave Equation: $\nabla^2 \underline{\mathbf{E}} = \mu\sigma_C \dfrac{\partial \underline{\mathbf{E}}}{\partial t} + \mu\varepsilon \dfrac{\partial^2 \underline{\mathbf{E}}}{\partial t^2}$

$\therefore \qquad -k^2 \underline{\mathbf{E}} = j\omega\mu\sigma_C \underline{\mathbf{E}} - \omega^2 \mu\varepsilon \underline{\mathbf{E}}$

or $\qquad k^2 = \omega^2 \mu\varepsilon - j\omega\mu\sigma_C \qquad (1)$

If we let $k = \alpha - j\beta$ then $k^2 = \alpha^2 - \beta^2 - 2j\alpha\beta \qquad (2)$

So using (1) and (2) we get

$$\alpha^2 - \beta^2 = \omega^2 \mu\varepsilon \qquad \text{(A)}$$

and

$$2\alpha\beta = \omega\mu\sigma_C \qquad \text{(B)}$$

Which we need to solve for α and β

Eqn. B gives us $\beta^2 = \dfrac{\omega^2\mu^2\sigma_C^2}{4\alpha^2}$, which we can use with Eqn. A to get

$$\alpha^2 - \frac{\omega^2\mu^2\sigma_C^2}{4\alpha^2} = \omega^2\mu\varepsilon \quad \Rightarrow \quad \alpha^4 - \omega^2\mu\varepsilon\alpha^2 - \frac{\omega^2\mu^2\sigma_C^2}{4} = 0$$

$\alpha^2 = \dfrac{\omega^2\mu\varepsilon \pm \sqrt{\omega^4\mu^2\varepsilon^2 + \omega^2\mu^2\sigma_C^2}}{2}$. Using $\alpha^2 - \beta^2 = \omega^2\mu\varepsilon$ and $c^2 = \dfrac{1}{\mu_0\varepsilon_0}$

$$\Rightarrow \qquad \alpha^2 = \frac{\omega^2\mu_r}{2c^2}\left[+\varepsilon_r + \sqrt{\varepsilon_r^{\,2} + \left(\frac{\sigma_C}{\varepsilon_0\omega}\right)^2}\,\right]$$

$$\Rightarrow \qquad \beta^2 = \frac{\omega^2\mu_r}{2c^2}\left[-\varepsilon_r + \sqrt{\varepsilon_r^{\,2} + \left(\frac{\sigma_C}{\varepsilon_0\omega}\right)^2}\,\right]$$

Note $\sigma_{C} \to 0$ $\beta^2 \to 0$ and $\alpha^2 \to \omega^2 \mu\varepsilon$

Note $\dfrac{\sigma_{C}}{\varepsilon\omega} >> 1$ $\alpha^2 = \beta^2 = \dfrac{\omega\mu_r\sigma_{C}}{2\varepsilon_0 c^2}$

We saw that for $\underline{\mathbf{E}} = \underline{\mathbf{j}}\tilde{E}_{y0}\exp[-\beta x]\exp[j(\omega t - \alpha x)]$

we got $\underline{\mathbf{H}} = \underline{\mathbf{k}}\dfrac{\sqrt{\alpha^2+\beta^2}}{\mu\omega}\tilde{E}_{y0}\exp[-\beta x]\exp[j(\omega t - \alpha x - \phi)]$

where we have written $\alpha - j\beta = \sqrt{\alpha^2+\beta^2}\exp(-j\phi)$ and $\tan\phi = \dfrac{\beta}{\alpha}$

<u>Refractive Index</u>

We should also notice that in a conductor $(\sigma_{C} \neq 0)$ the refractive index is complex

$$k = \frac{\omega n}{c} \quad \Rightarrow \quad n^2 = \frac{c^2k^2}{\omega^2} = \frac{c^2}{\omega^2}\left(\omega^2\mu\varepsilon - j\omega\mu\sigma_C\right) = \frac{c^2}{\omega^2}\left(\alpha^2 - \beta^2 - 2j\alpha\beta\right)$$

$$n^2 = \mu_r\left(\varepsilon_r - j\frac{\sigma_C}{\varepsilon_0\omega}\right)$$

So we see that a complex k means we must have a complex n.

So we could write $n = n_\alpha - jn_\beta$ and equate with $k = \alpha - j\beta$ giving

$$\alpha = \frac{\omega n_\alpha}{c} \qquad \text{and} \qquad \beta = \frac{\omega n_\beta}{c}$$

Now $\dfrac{\omega n_\alpha}{c} = \dfrac{2\pi}{\lambda_0}n_\alpha = \dfrac{\omega}{v} = \dfrac{2\pi}{\lambda}$ so that we can write $\lambda = \dfrac{\lambda_0}{n_\alpha}$ and $v = \dfrac{c}{n_\alpha}$

The wavelength λ and the velocity v in the medium are determined from the free space values by n_α or equivalently α.

The Range of the wave in the medium is determined by β or equivalently n_β.

THE SKIN EFFECT IN GOOD CONDUCTORS: $\omega\mu\sigma_C >> \omega^2\mu\varepsilon$

$\alpha^2 = \beta^2 = \dfrac{\omega\mu\sigma_C}{2}$ and the "skin depth" $\delta = \dfrac{1}{\beta} = \sqrt{\dfrac{2}{\omega\mu\sigma_C}}$

We can now re-write the $\underline{\mathbf{E}}$ field we are using in the previous example as

$$\underline{\mathbf{E}} = \underline{\mathbf{j}}\tilde{E}_{y0}\exp\left[-\frac{x}{\delta}\right]\exp\left[j\left(\omega t - \frac{x}{\delta}\right)\right]$$

Since $\underline{\mathbf{J}} = \sigma_C\underline{\mathbf{E}}$

$$\underline{\mathbf{J}} = \underline{\mathbf{j}}\tilde{J}_{y0}\exp\left[-\frac{x}{\delta}\right]\exp\left[j\left(\omega t - \frac{x}{\delta}\right)\right]$$

The $\underline{\mathbf{E}}$ field causes a current to flow in the conductor and the wave energy is dissipated in a few skin depths as Joule Heating and both $\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ fields decay to zero. (What happens in a perfect conductor?)

Now consider the flow of current down a wire of circular section and radius a.

In the D.C. case $\omega \to 0$ the current density is uniform and $I = J\pi a^2$.

But at high frequencies practically all the current flow is confined to a thin "sheet" or "skin" at the surface. The maths to show this involves Bessel functions – but if $a >> \delta$ then we can find

$$|\underline{\mathbf{J}}| = J = J_{r=a}\exp\left[-\frac{(a-r)}{\delta}\right]$$

If $a >> \delta$ 95% of the current flows within 3δ of the surface $\Rightarrow$ hollow conductors are as good as solid ones! Note as $\delta \propto \omega^{-0.5}$ the higher the frequency the thinner the skin, which implies the higher the resistance since effective area gets smaller as ω increases.

ENERGY TRANSFER BY AN ELECTROMAGNETIC FIELD

THE POYNTING VECTOR

Start with the Maxwell curl equations:

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + \frac{\partial \underline{\mathbf{D}}}{\partial t} \qquad \nabla \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

$$\underline{\mathbf{E}} \bullet \nabla \times \underline{\mathbf{H}} = \underline{\mathbf{E}} \bullet \underline{\mathbf{J}} + \underline{\mathbf{E}} \bullet \frac{\partial \underline{\mathbf{D}}}{\partial t} \qquad \underline{\mathbf{H}} \bullet \nabla \times \underline{\mathbf{E}} = -\underline{\mathbf{H}} \bullet \frac{\partial \underline{\mathbf{B}}}{\partial t}$$

$$\underline{\mathbf{H}} \bullet \nabla \times \underline{\mathbf{E}} - \underline{\mathbf{E}} \bullet \nabla \times \underline{\mathbf{H}} = -\underline{\mathbf{E}} \bullet \underline{\mathbf{J}} - \left(\underline{\mathbf{E}} \bullet \frac{\partial \underline{\mathbf{D}}}{\partial t} + \underline{\mathbf{H}} \bullet \frac{\partial \underline{\mathbf{B}}}{\partial t} \right)$$

For any two vectors $\nabla \bullet \underline{\mathbf{X}} \times \underline{\mathbf{Y}} = \underline{\mathbf{Y}} \bullet \nabla \times \underline{\mathbf{X}} - \underline{\mathbf{X}} \bullet \nabla \times \underline{\mathbf{Y}}$

Therefore $\nabla \bullet \underline{\mathbf{E}} \times \underline{\mathbf{H}} = -\underline{\mathbf{E}} \bullet \underline{\mathbf{J}} - \left(\underline{\mathbf{E}} \bullet \frac{\partial \underline{\mathbf{D}}}{\partial t} + \underline{\mathbf{H}} \bullet \frac{\partial \underline{\mathbf{B}}}{\partial t} \right)$

Consider volume v bounded by a surface S and integrate over the volume

$$\int_{\mathrm{v}} \nabla \bullet \underline{\mathbf{E}} \times \underline{\mathbf{H}}\, \mathrm{dv} = -\int_{\mathrm{v}} \underline{\mathbf{E}} \bullet \underline{\mathbf{J}}\, \mathrm{dv} - \int_{\mathrm{v}} \left(\underline{\mathbf{E}} \bullet \frac{\partial \underline{\mathbf{D}}}{\partial t} + \underline{\mathbf{H}} \bullet \frac{\partial \underline{\mathbf{B}}}{\partial t} \right) \mathrm{dv}$$

$\underline{\mathbf{\Pi}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}$ POYNTING VECTOR [Vm^{-1}][Am^{-1}]=[Wm^{-2}]

Gauss' Theorem $\int_{\mathrm{v}} \nabla \bullet \underline{\mathbf{\Pi}}\, \mathrm{dv} = \oint_{\mathrm{S}} \underline{\mathbf{\Pi}} \bullet \mathrm{d}\underline{\mathbf{S}}$

$$\oint_{\mathrm{S}} \underline{\mathbf{\Pi}} \bullet \mathrm{d}\underline{\mathbf{S}} + \int_{\mathrm{v}} \underline{\mathbf{E}} \bullet \underline{\mathbf{J}}\, \mathrm{dv} = -\int_{\mathrm{v}} \left(\underline{\mathbf{E}} \bullet \frac{\partial \underline{\mathbf{D}}}{\partial t} + \underline{\mathbf{H}} \bullet \frac{\partial \underline{\mathbf{B}}}{\partial t} \right) \mathrm{dv}$$

POYNTING'S THEOREM

For linear and isotropic media: $\underline{\mathbf{D}} = \varepsilon \underline{\mathbf{E}}$ and $\underline{\mathbf{B}} = \mu \underline{\mathbf{H}}$

$$\oint_{\mathrm{S}} \underline{\mathbf{\Pi}} \bullet \mathrm{d}\underline{\mathbf{S}} + \int_{\mathrm{v}} \underline{\mathbf{E}} \bullet \underline{\mathbf{J}}\, \mathrm{dv} = -\frac{\partial}{\partial t} \int_{\mathrm{v}} \left(\frac{\varepsilon \mathrm{E}^2}{2} + \frac{\mu \mathrm{H}^2}{2} \right) \mathrm{dv}$$

Interpretation of Poynting's Theorem when the volume does not contain a power source.

$$\oint_S \underline{\mathbf{\Pi}} \bullet d\underline{\mathbf{S}} \quad + \quad \int_v \underline{\mathbf{E}} \bullet \underline{\mathbf{J}}\, dv \quad = \quad -\frac{\partial}{\partial t}\int_v \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv$$

By conservation of energy this must be the rate at which EM energy is leaving v through the surface S

$\Rightarrow$ Power flow through S [Js^{-1}=W]

$$= \int_v \frac{J^2}{\sigma_C} dv = \int_v J^2 \rho_R \, dv$$

(using $\underline{\mathbf{J}} = \sigma_C \underline{\mathbf{E}}$)

EM energy is dissipated as Joule Heating within the volume. [Js^{-1}=W]

Sum of the Electric and Magnetic energies in the volume (as derived in statics) =

$$\int_v \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv$$

$\therefore$

$$-\frac{\partial}{\partial t}\int_v \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv$$

$= -1\times$(Rate of change of EM energy in the volume).
= Rate of decrease of EM energy in the volume.
[Js^{-1}=W]

NOTE (1) The EM energy within the volume decreases because some is converted into Joule Heating of the medium and the rest is leaving the volume through its surface.

NOTE (2) If $\Rightarrow \sigma_C = 0$. No Joule heating and all of power flowing through the surface

$$\oint_S \underline{\mathbf{\Pi}} \bullet d\underline{\mathbf{S}} = -\frac{\partial}{\partial t}\int_v \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv.$$

NOTE (3) $\underline{\mathbf{\Pi}}$ = Flux of power through the surface.

Interpretation of Poynting's Theorem when the volume contains a power source.

What is the power source?

Eg. 1. Inside a battery an electric field $\underline{\mathbf{E}}'$ is produced by an electro-chemical reaction – work done by chemical reaction.

Eg. 2. Within a dynamo an electric field $\underline{\mathbf{E}}'$ is produced by electromagnetic induction – mechanical work done.

The current density at any point is now given by

$$\underline{\mathbf{J}} = \sigma_C\left(\underline{\mathbf{E}} + \underline{\mathbf{E}}'\right) \qquad \Rightarrow \underline{\mathbf{E}} = \frac{\underline{\mathbf{J}}}{\sigma_C} - \underline{\mathbf{E}}'$$

$$\therefore \int_v \underline{\mathbf{E}} \bullet \underline{\mathbf{J}}\, dv = \int_v \frac{J^2}{\sigma_C} dv - \int_v \underline{\mathbf{E}}' \bullet \underline{\mathbf{J}}\, dv$$

As before this term represents EM energy dissipated through Joule Heating within the volume. [Js^{-1}=W]

Power generated within the volume [Js^{-1}=W] P_G

$$\therefore \text{Now } P_G = \oint_S \underline{\mathbf{\Pi}} \bullet d\underline{\mathbf{S}} \quad + \quad \int_v \frac{J^2}{\sigma_C} dv \quad + \quad \frac{\partial}{\partial t}\int_v \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv$$

Part of Power generated within the volume leaves through surface.

Part of Power is dissipated within the volume as Joule heating.

The remainder increases the EM energy contained in V

Note in ideal dielectric $\underline{\mathbf{J}} = 0$. In steady state $\frac{\partial}{\partial t}(\ ) \rightarrow 0$, $\therefore$ all power generated flows through the surface so $P_G = \oint_S \underline{\mathbf{\Pi}} \bullet d\underline{\mathbf{S}}$

Applications of Poynting's theorem

Linearly Polarised Plane Wave in an ideal dielectric

$$\underline{\mathbf{E}} = \underline{\mathbf{j}}E_y \qquad E_y = E_0 \cos(\omega t - kx)$$

$$\frac{E_y}{H_z} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\underline{\mathbf{H}} = \underline{\mathbf{k}}H_z \qquad H_z = \sqrt{\frac{\varepsilon}{\mu}} E_0 \cos(\omega t - kx)$$

$$\underline{\mathbf{\Pi}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}} = \underline{i} E_y H_z$$

$$\Pi_x = \sqrt{\frac{\varepsilon}{\mu}} E_0^2 \cos^2(\omega t - kx)$$ [Note always positive]

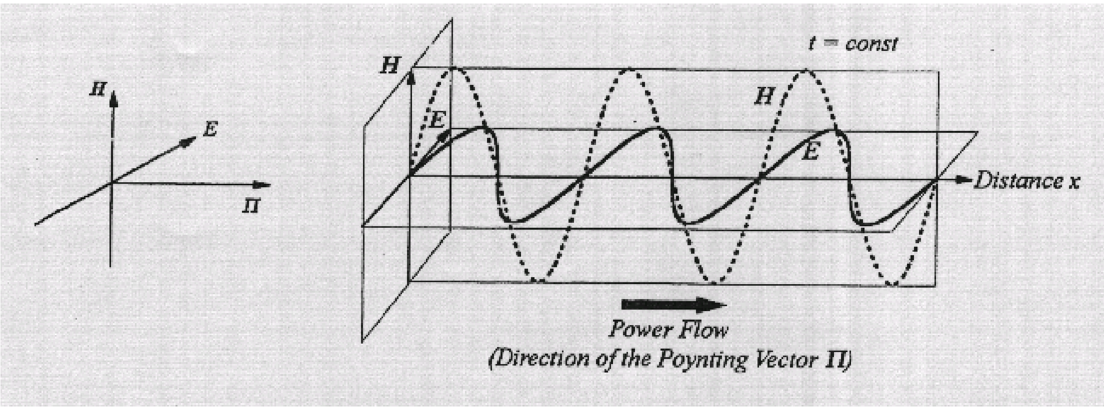
$\therefore$ Time average Poynting Vector $\langle \Pi_x \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}} E_0^2 = \sqrt{\frac{\varepsilon}{\mu}} E_{RMS}^2$

NOTE: $$\begin{bmatrix} \text{Mean rate of energy flow} \\ \text{per unit area} \end{bmatrix} = \begin{bmatrix} \text{Mean energy} \\ \text{density} \end{bmatrix} \times \begin{bmatrix} \text{Velocity} \\ \text{of flow} \end{bmatrix}$$

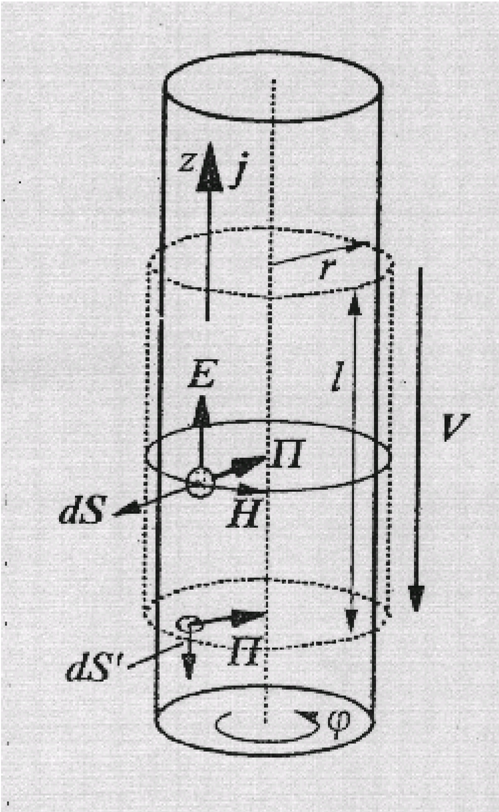
$$\langle \Pi_x \rangle \qquad = \qquad \langle W \rangle \qquad \times \qquad [v_e]$$

$$\langle W \rangle = \frac{1}{2}\left\langle \varepsilon E^2 + \mu H^2 \right\rangle = \frac{1}{4}\left[\varepsilon E_0{}^2 + \mu H_0{}^2\right] = \frac{1}{4}\left[\varepsilon E_0{}^2 + \varepsilon E_0{}^2\right] = \frac{\varepsilon E_0{}^2}{2}$$

$$\therefore [v_e] = \frac{\langle \Pi_x \rangle}{\langle W \rangle} = \frac{1}{\sqrt{\mu\varepsilon}}$$ [Hurray!]



Power Dissipation in a wire (constant current)



Consider a circular wire radius a carrying a current I.

If V is the P.D. dropped between z_0 and $z_0 + l$

$$-\int_{z_0}^{z_0+l} E_z \mathrm{dz} = -V \qquad E_z = \frac{V}{l}$$

[Note for all $r < a$, $E_z = \frac{V}{l}$ but for $r > a$ $E_z = 0$]

At $r \geq a$ Amperes Law gives

$$H_\varphi = \frac{I}{2\pi r}$$

$\underline{\mathbf{E}} = \hat{\underline{\mathbf{z}}} E_z$ and $\underline{\mathbf{H}} = \hat{\underline{\boldsymbol{\varphi}}} H_\varphi$ so $\underline{\mathbf{\Pi}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}} = \hat{\underline{\mathbf{z}}} E_z \times \hat{\underline{\boldsymbol{\varphi}}} H_\varphi$

$$\underline{\mathbf{\Pi}} = \frac{V}{l}\frac{I}{2\pi r}\hat{\underline{\mathbf{z}}} \times \hat{\underline{\boldsymbol{\varphi}}} = \frac{VI}{l2\pi r}\left(-\hat{\underline{\mathbf{r}}}\right)$$

Now consider integration over surface of wire:

$$-\oint_S \underline{\mathbf{\Pi}} \bullet \mathrm{d}\underline{\mathbf{S}} = -\oint_S \frac{VI}{l2\pi a}\left(-\hat{\underline{\mathbf{r}}}\right) \bullet \mathrm{d}\underline{\mathbf{S}} = \frac{VI}{l2\pi a} l2\pi a = VI$$

$\oint_S \underline{\mathbf{\Pi}} \bullet \mathrm{d}\underline{\mathbf{S}}$ is negative i.e. power flows inwards from the surface.

i.e. The power dissipated in the wire is a result of an “inflow” of power associated with fields of the wire through its surface.

Note in this case the $\frac{\partial}{\partial t}$ term in the Poynting’s theorem is zero

$$-\oint_S \underline{\mathbf{\Pi}} \bullet \mathrm{d}\underline{\mathbf{S}} = \int_v \underline{\mathbf{E}} \bullet \underline{\mathbf{J}}\, \mathrm{dv} = \frac{V}{l}\frac{I}{\pi a^2}\pi a^2 l = VI$$

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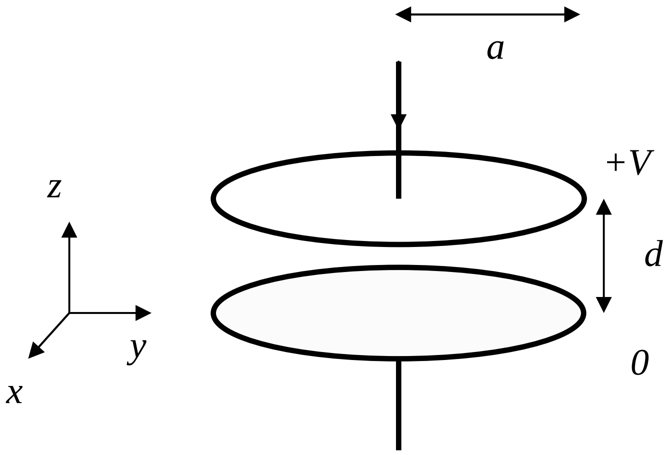
Consider a constant current I flowing to increase the charge on the plates of a capacitor.



For $r \le a$ we get $\underline{\mathbf{E}} = \frac{V}{d}\left(-\hat{\underline{z}}\right)$

Applying Ampere's Law: $\oint_C \underline{\mathbf{H}} \bullet \mathrm{d}\underline{\mathbf{l}} = \int_S \frac{\partial \underline{\mathbf{D}}}{\partial t}\,\mathrm{d}\underline{\mathbf{S}} = -I$. $r \ge a$

For $r \ge a$ we get $\underline{\mathbf{H}} = \left(-\hat{\underline{\boldsymbol{\varphi}}}\right)\frac{I}{2\pi r}$

$$\underline{\mathbf{\Pi}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}} = \frac{V}{d}\left(-\hat{\underline{z}}\right)\times\left(-\hat{\underline{\boldsymbol{\varphi}}}\right)\frac{I}{2\pi r}$$

$$\underline{\mathbf{\Pi}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}} = \frac{IV}{2\pi rd}\left(-\hat{\underline{\mathbf{r}}}\right)$$

Now consider integration over surface at edge of capacitor

$$-\oint_S \underline{\mathbf{\Pi}} \bullet \mathrm{d}\underline{\mathbf{S}} = -\oint_S \frac{VI}{2\pi ad}\left(-\hat{\underline{\mathbf{r}}}\right)\bullet \mathrm{d}\underline{\mathbf{S}} = \frac{VI}{2\pi ad}\, d\, 2\pi a = VI$$

"Inflow" of power associated with fields at the surface. In between the plates conductivity is zero – no Joule Heating. Where is this power going?

$$-\oint_S \underline{\mathbf{\Pi}} \bullet \mathrm{d}\underline{\mathbf{S}} = \frac{\partial}{\partial t}\int_v \left(\frac{\varepsilon \mathrm{E}^2}{2} + \frac{\mu \mathrm{H}^2}{2}\right)\mathrm{dv}$$

Electromagnetic Momentum Density : $\underline{\mathbf{G}}$

$\underline{\mathbf{G}} = \underline{\mathbf{D}} \times \underline{\mathbf{B}}$ [Cm^{-2}Tesla] = [Cm^{-2}NA^{-1}m^{-1}] = [Asm^{-2}NA^{-1}m^{-1}]=[Ns m^{-3}]

For linear media

$$\underline{\mathbf{G}} = \underline{\mathbf{D}} \times \underline{\mathbf{B}} = \varepsilon\underline{\mathbf{E}} \times \mu\underline{\mathbf{H}} = \varepsilon\mu\underline{\mathbf{E}} \times \underline{\mathbf{H}} = \varepsilon\mu\underline{\mathbf{\Pi}}$$

$$\underline{\mathbf{G}} = \frac{\underline{\mathbf{\Pi}}}{v^2}$$

Radiation Pressure

Consider an EM wave propagating with velocity $v = \dfrac{1}{\sqrt{\mu\varepsilon}}$ in a linear medium.

If the wave is incident normally on a totally absorbing surface, then in one second the momentum absorbed per unit area of the surface $= \underline{v}.\underline{\mathbf{G}}$

Therefore, $P_r = \dfrac{|\underline{\mathbf{\Pi}}|}{v}$ [Nm^{-2}]

If the surface is perfectly reflecting $P_r = \dfrac{2|\underline{\mathbf{\Pi}}|}{v}$

The Poynting vector and the complex field notation.

Suppose that $\underline{\mathbf{E}} = \underline{\tilde{\mathbf{E}}}_0 \exp[j(\omega t - kx)]$

and $\underline{\mathbf{H}} = \underline{\tilde{\mathbf{H}}}_0 \exp[j(\omega t - kx)]$

Then $\underline{\mathbf{E}} \times \underline{\mathbf{H}} \propto \exp(j2\omega t)$ - time average over one period is zero!

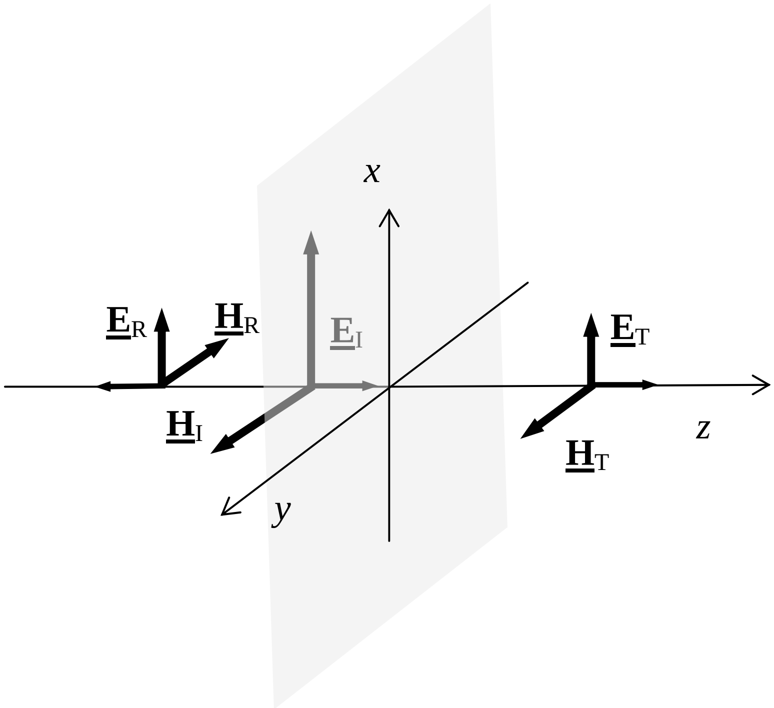
$$\underline{\mathbf{\Pi}} = \mathrm{Re}(\underline{\mathbf{E}}) \times \mathrm{Re}(\underline{\mathbf{H}})$$

Reflection and the Fresnel Equations

"Books are like a mirror. If an ass looks in, you can't expect an angel to look out." B C Forbes

Reflection of Plane Wave at a Dielectric Boundary (incident on dielectric from free space).

(1) Normal Incidence



$$\underline{\mathbf{E}}_{\mathrm{I}} = \underline{\mathbf{i}} E_{0I} \exp\left[j\left(\omega t - k_1 z\right)\right]$$

$$\underline{\mathbf{H}}_{\mathrm{I}} = \underline{\mathbf{j}} H_{0I} \exp\left[j\left(\omega t - k_1 z\right)\right]$$

$$k_1 = \omega\sqrt{\mu_0 \varepsilon_0}$$

$$\underline{\mathbf{E}}_{\mathrm{R}} = \underline{\mathbf{i}} E_{0R} \exp\left[j\left(\omega t + k_1 z\right)\right]$$

$$\underline{\mathbf{H}}_{\mathrm{R}} = -\underline{\mathbf{j}} H_{0R} \exp\left[j\left(\omega t + k_1 z\right)\right]$$

$$\underline{\mathbf{E}}_{\mathrm{T}} = \underline{\mathbf{i}} E_{0\mathrm{T}} \exp\left[j\left(\omega t - k_2 z\right)\right]$$

$$\underline{\mathbf{H}}_{\mathrm{T}} = \underline{\mathbf{j}} H_{0\mathrm{T}} \exp\left[j\left(\omega t - k_2 z\right)\right]$$

$$k_2 = \omega\sqrt{\mu\varepsilon}$$

Place boundary at $z = 0$ and use boundary conditions.
Tangential component of E-field must be continuous at boundary.

$$E_{0\mathrm{I}} + E_{0\mathrm{R}} = E_{0\mathrm{T}} \qquad \text{(A1)}$$

Tangential component of H-field must be continuous at boundary (so long as no surface currents).

$$H_{0\mathrm{I}} - H_{0\mathrm{R}} = H_{0\mathrm{T}}$$

We know that $\dfrac{E_{0\mathrm{I}}}{H_{0\mathrm{I}}} = \sqrt{\dfrac{\mu}{\varepsilon}}$ etc $\Rightarrow \sqrt{\dfrac{\varepsilon_0}{\mu_0}} E_{0\mathrm{I}} - \sqrt{\dfrac{\varepsilon_0}{\mu_0}} E_{0\mathrm{R}} = \sqrt{\dfrac{\varepsilon}{\mu}} E_{0\mathrm{T}}$

$$\Rightarrow E_{0\mathrm{I}} - E_{0\mathrm{R}} = \sqrt{\mu_r \varepsilon_r} E_{0\mathrm{T}} \Rightarrow \qquad E_{0\mathrm{I}} - E_{0\mathrm{R}} = nE_{0\mathrm{T}} \qquad \text{(B1)}$$

[Assuming $\mu_r = 1$]

Using (A1) and (B1) $t_n = \frac{E_{0\mathrm{T}}}{E_{0\mathrm{I}}} = \frac{2}{1+n}$ and $r_n = \frac{E_{0\mathrm{R}}}{E_{0\mathrm{I}}} = \frac{1-n}{1+n}$

Reflection Coefficient = Reflected energy/Incident energy

$$= \frac{\text{Time averaged reflected Poynting vector}}{\text{Time averaged incident Poynting vector}}$$

$$R = \left(\frac{E_{0\mathrm{R}}}{E_{0\mathrm{I}}}\right)^2 = \left(\frac{1-n}{1+n}\right)^2$$

Transmission Coefficient = $1-R$

$$T = \frac{4n}{(1+n)^2} = \frac{\frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}E_{0\mathrm{T}}^2}{\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}E_{0\mathrm{I}}^2} = \frac{\text{Time averaged transmitted Poynting vector}}{\text{Time averaged incident Poynting vector}}$$

(2) **E** field perpendicular to the plane of incidence

Tangential component of E-field must be continuous at boundary.

$$E_{0\mathrm{I}} + E_{0\mathrm{R}} = E_{0\mathrm{T}} \qquad \text{(A2)}$$

Tangential component of H-field must be continuous at boundary (so long as no surface currents).

$$-H_{0\mathrm{I}}\cos\theta_I + H_{0\mathrm{R}}\cos\theta_R = -H_{0\mathrm{T}}\cos\theta_T$$

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}E_{0\mathrm{I}}\cos\theta_I - \sqrt{\frac{\varepsilon_0}{\mu_0}}E_{0\mathrm{R}}\cos\theta_R = \sqrt{\frac{\varepsilon}{\mu}}E_{0\mathrm{T}}\cos\theta_T$$

$$E_{0\mathrm{I}}\cos\theta_I - E_{0\mathrm{R}}\cos\theta_R = nE_{0\mathrm{T}}\cos\theta_T$$

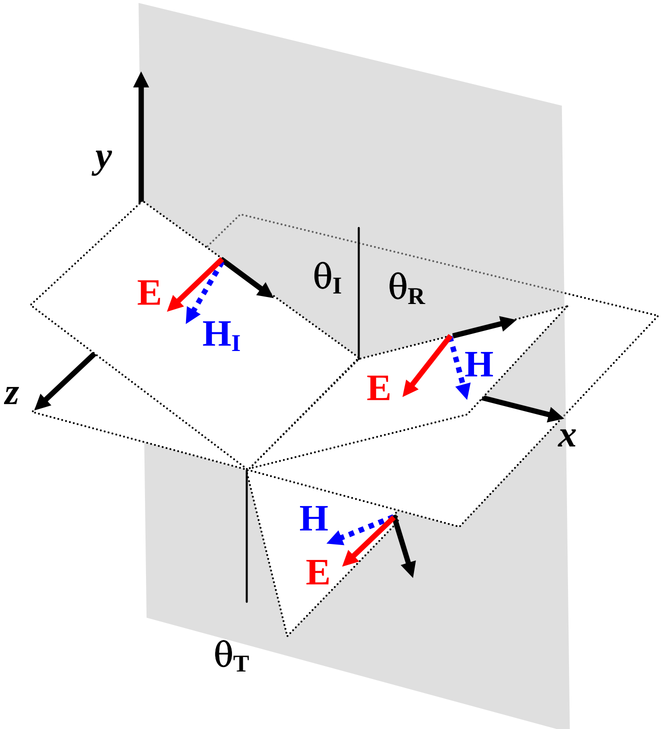
[Assuming $\mu_r = 1$]
and using $\cos\theta_I = \cos\theta_R$

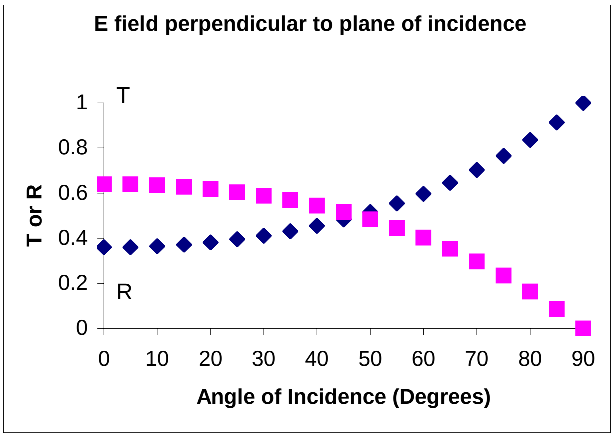
$$E_{0\mathrm{I}}\cos\theta_I - E_{0\mathrm{R}}\cos\theta_I = nE_{0\mathrm{T}}\cos\theta_T \qquad \text{(B2)}$$

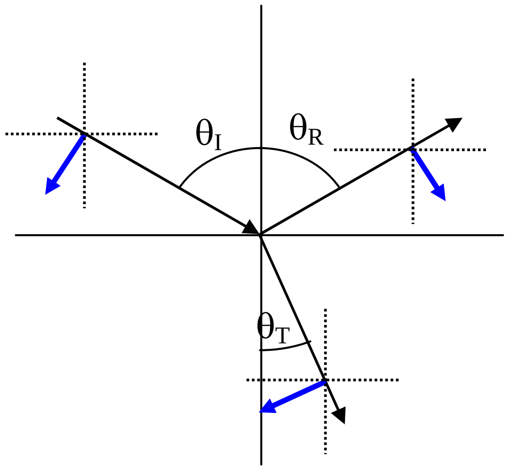
Using (A2) and (B2)

$$t_\perp = \frac{E_{0\mathrm{T}}}{E_{0\mathrm{I}}} = \frac{2\cos\theta_I}{\cos\theta_I + n\cos\theta_T}$$

$$r_\perp = \frac{E_{0\mathrm{R}}}{E_{0\mathrm{I}}} = \frac{\cos\theta_I - n\cos\theta_T}{\cos\theta_I + n\cos\theta_T}$$







(2) **E** field parallel to the plane of incidence

Tangential component of E-field must be continuous at boundary.

$$E_{0\mathrm{I}}\cos\theta_I - E_{0\mathrm{R}}\cos\theta_R = E_{0\mathrm{T}}\cos\theta_T \qquad \text{(A3)}$$

Tangential component of H-field must be continuous at boundary (so long as no surface currents).

$$H_{0\mathrm{I}} + H_{0\mathrm{R}} = H_{0\mathrm{T}}$$

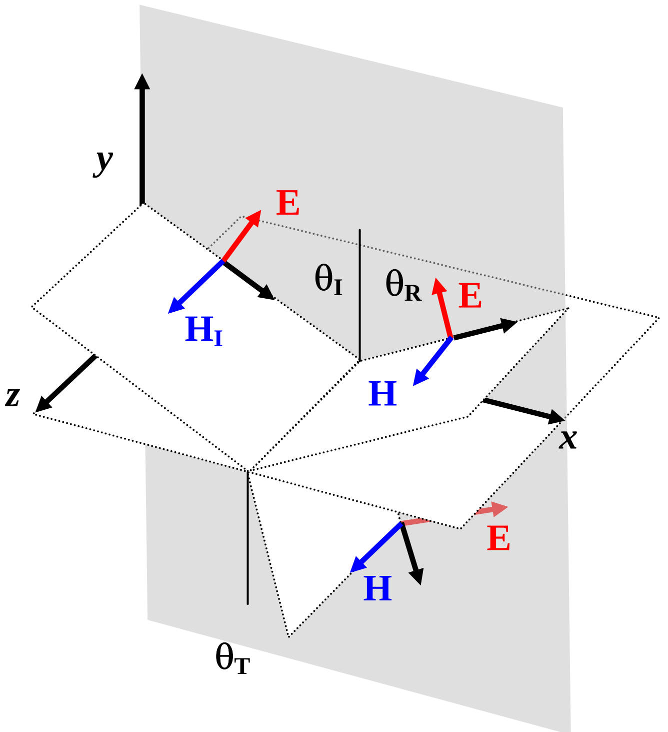
[Assuming $\mu_r = 1$]
and using $\cos\theta_I = \cos\theta_R$

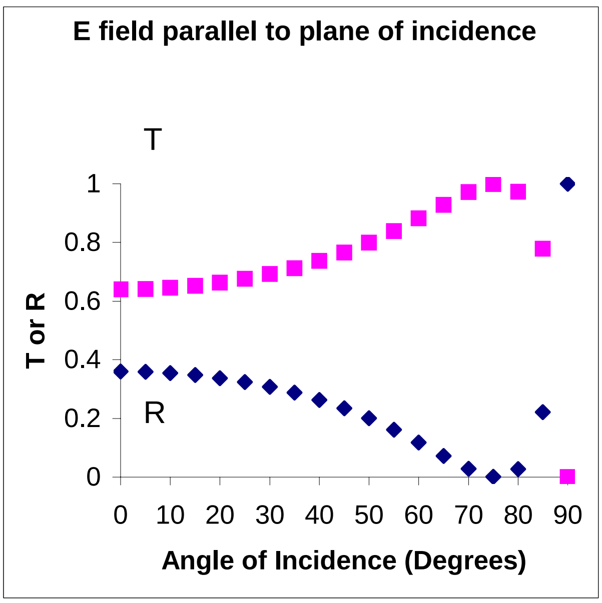
$$E_{0\mathrm{I}} + E_{0\mathrm{R}} = nE_{0\mathrm{T}} \qquad \text{(B3)}$$

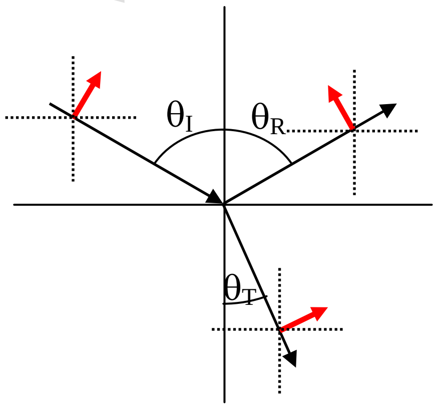
Using (A3) and (B3)

$$t_{\parallel} = \frac{E_{0\mathrm{T}}}{E_{0\mathrm{I}}} = \frac{2\cos\theta_I}{n\cos\theta_I + \cos\theta_T}$$

$$r_{\parallel} = \frac{E_{0\mathrm{R}}}{E_{0\mathrm{I}}} = \frac{n\cos\theta_I - \cos\theta_T}{n\cos\theta_I + \cos\theta_T}$$







Note 1: The formula given above are called the “Fresnel Equations”.

Note 2: The formula above can be simplified using Snell’s Law
$n_1 \sin\theta_1 = n_2 \sin\theta_2$, or in our derivations $\sin\theta_I = n\sin\theta_T$

$$r_\perp = -\frac{\sin(\theta_I - \theta_T)}{\sin(\theta_I + \theta_T)} \qquad r_\parallel = \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)}$$

$$t_\perp = \frac{2\sin\theta_T \cos\theta_I}{\sin(\theta_I + \theta_T)} \qquad t_\parallel = \frac{2\sin\theta_T \cos\theta_I}{\sin(\theta_I + \theta_T)\cos(\theta_I - \theta_T)}$$

We see that $r_\parallel \to 0$ as $\theta_I + \theta_T \to \frac{\pi}{2}$ so no light is reflected for this polarisation.

Note also that $r_\parallel$ changes sign at $\theta_I + \theta_T = \frac{\pi}{2}$ so there is a phase shift of π in the reflected parallel components of the E- (and H-) fields when sweeping the incident angle θ_I through the polarisation angle θ_P (θ_P is the value of θ_I for which $\theta_I + \theta_T = \frac{\pi}{2}$). However, $r_\perp$ is always negative so no phase change in the reflected perpendicular components of the E- (and H-) fields.

Note 3: Be careful using the Fresnel Equations, must get the polarisation correct! Easy to get confused!

Note 4: Can calculate the reflected and transmitted intensities using the Fresnel Equations.

Note 5: Could derive the Fresnel Equations for transmission a boundary where $n_I > n_T$ and investigate total internal reflection…but not here! See for example Optics (Second Edition) by Hecht.

Reflection at a conducting boundary.

Consider the case where plane wave travelling in free space strikes at normal incidence a conducting boundary

$$\underline{\mathbf{E}}_{\mathrm{I}} = \underline{\mathbf{i}}E_{0I}\exp\left[j(\omega t - k_1 z)\right] \qquad k_1 = \omega\sqrt{\mu_0\varepsilon_0}$$
$$\underline{\mathbf{H}}_{\mathrm{I}} = \underline{\mathbf{j}}H_{0I}\exp\left[j(\omega t - k_1 z)\right]$$

$$\underline{\mathbf{E}}_{\mathrm{R}} = \underline{\mathbf{i}}E_{0R}\exp\left[j(\omega t + k_1 z)\right]$$

$$\underline{\mathbf{H}}_{\mathrm{R}} = -\underline{\mathbf{j}}H_{0R}\exp\left[j(\omega t + k_1 z)\right]$$

$$\underline{\mathbf{E}}_{T} = \underline{\mathbf{i}}E_{0T}\exp[-\beta z]\exp\left[j(\omega t - \alpha z)\right] \qquad k_2 = \alpha - j\beta$$

$$\underline{\mathbf{H}}_{T} = \underline{\mathbf{j}}H_{0T}\exp[-\beta z]\exp\left[j(\omega t - kz)\right]$$
$$\underline{\mathbf{H}}_{T} = \underline{\mathbf{j}}\frac{(\alpha - j\beta)}{\mu\omega}E_{0T}\exp[-\beta z]\exp\left[j(\omega t - kz)\right] \qquad \text{(See lecture 12)}$$

Boundary at $z = 0$. Tangential component of E-field must be continuous at boundary.

$$E_{0\mathrm{I}} + E_{0\mathrm{R}} = E_{0\mathrm{T}} \qquad \text{(A4)}$$

Tangential component of H-field must be continuous at boundary (so long as no surface current unit length flowing on the boundary, i.e. we have a good conductor not a perfect conductor).

$$H_{0\mathrm{I}} - H_{0\mathrm{R}} = H_{0\mathrm{T}}$$

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}E_{0\mathrm{I}} - \sqrt{\frac{\varepsilon_0}{\mu_0}}E_{0\mathrm{R}} = \frac{(\alpha - j\beta)}{\mu\omega}E_{0T}$$

For a good conductor $\alpha = \beta = \sqrt{\dfrac{\omega\mu\sigma_C}{2}}$

$$E_{0\mathrm{I}} - E_{0\mathrm{R}} = (1 - j)\sqrt{\frac{\sigma_C}{2\omega\mu_r\varepsilon_0}}E_{0T} \qquad \text{(B4)}$$

Using (A4) and (B4)

$$r_n = \frac{E_{0\mathrm{R}}}{E_{0\mathrm{I}}} = \frac{1-(1-j)\sqrt{\dfrac{\sigma_C}{2\omega\mu_r\varepsilon_0}}}{1+(1-j)\sqrt{\dfrac{\sigma_C}{2\omega\mu_r\varepsilon_0}}}$$

$$R_n = \left(\frac{E_{0\mathrm{R}}}{E_{0\mathrm{I}}}\right)^2 \approx 1-2\sqrt{\frac{2\omega\mu_r\varepsilon_0}{\sigma_C}}$$

For copper at infrared frequencies (around 10^{14} Hz) $\sqrt{\dfrac{2\omega\mu_r\varepsilon_0}{\sigma_C}} \approx 0.01$, so about 98% of infrared radiation is reflected, the remainder is absorbed in the metal due to Joule Heating.

At lower frequencies (e.g. radio waves) $\sqrt{\dfrac{2\omega\mu_r\varepsilon_0}{\sigma_C}} \approx 10^{-6}$ almost all radiation is reflected.

At higher frequencies (>10^{15} Hz), simple theory does not work, we need to take account of the atomic transitions that take place and give rise to the colour of the metal.