Cartesian Coordinates (x,y,z):

$$\nabla\Psi = \hat{x}\frac{\partial\Psi}{\partial x} + \hat{y}\frac{\partial\Psi}{\partial y} + \hat{z}\frac{\partial\Psi}{\partial z}$$

$$\nabla\bullet\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\overline{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}$$

Cylindrical coordinates (r,ϕ,z):

$$\nabla\Psi = \hat{r}\frac{\partial\Psi}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial\Psi}{\partial\phi} + \hat{z}\frac{\partial\Psi}{\partial z}$$

$$\nabla\bullet\overline{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\overline{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right) = \frac{1}{r}\det\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z\\ A_r & rA_\phi & A_z\end{vmatrix}$$

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{\partial^2\Psi}{\partial z^2}$$

Spherical coordinates (r,θ,ϕ):

$$\nabla\Psi = \hat{r}\frac{\partial\Psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}$$

$$\nabla\bullet\overline{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\overline{A} = \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

$$= \frac{1}{r^2\sin\theta}\det\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi}\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

Gauss' Divergence Theorem:

$$\int_V \nabla\bullet\overline{G}\,dv = \oint_A \overline{G}\bullet\hat{n}\,da$$

Stokes' Theorem:

$$\int_A (\nabla\times\overline{G})\bullet\hat{n}\,da = \oint_C \overline{G}\bullet d\ell$$

Vector Algebra:

$$\nabla = \hat{x}\partial/\partial x + \hat{y}\partial/\partial y + \hat{z}\partial/\partial z$$

$$\overline{A}\bullet\overline{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\nabla\bullet(\nabla\times\overline{A}) = 0$$

$$\nabla\times(\nabla\times\overline{A}) = \nabla(\nabla\bullet\overline{A}) - \nabla^2\overline{A}$$

<u>Basic Equations for Electromagnetics and Applications</u>

Fundamentals

$\bar{f} = q(\bar{E} + \bar{v} \times \mu_o \bar{H})[N]$ (Force on point charge)

$\nabla \times \bar{E} = -\partial \bar{B}/\partial t$

$\oint_c \bar{E} \bullet d\bar{s} = -\frac{d}{dt}\int_A \bar{B} \bullet d\bar{a}$

$\nabla \times \bar{H} = \bar{J} + \partial \bar{D}/\partial t$

$\oint_c \bar{H} \bullet d\bar{s} = \int_A \bar{J} \bullet d\bar{a} + \frac{d}{dt}\int_A \bar{D} \bullet d\bar{a}$

$\nabla \bullet \bar{D} = \rho \rightarrow \int_A \bar{D} \bullet d\bar{a} = \int_V \rho dv$

$\nabla \bullet \bar{B} = 0 \rightarrow \int_A \bar{B} \bullet d\bar{a} = 0$

$\nabla \bullet \bar{J} = -\partial \rho/\partial t$

$\bar{E}$ = electric field (Vm^{-1})
$\bar{H}$ = magnetic field (Am^{-1})
$\bar{D}$ = electric displacement (Cm^{-2})
$\bar{B}$ = magnetic flux density (T)
Tesla (T) = Weber m^{-2} = 10,000 gauss
ρ = charge density (Cm^{-3})
$\bar{J}$ = current density (Am^{-2})
σ = conductivity (Siemens m^{-1})
$\bar{J}_s$ = surface current density (Am^{-1})
ρ_s = surface charge density (Cm^{-2})
$\varepsilon_o = 8.85 \times 10^{-12}\ Fm^{-1}$
$\mu_o = 4\pi \times 10^{-7}\ Hm^{-1}$
$c = (\varepsilon_o\mu_o)^{-0.5} \cong 3 \times 10^8\ ms^{-1}$
$e = -1.60 \times 10^{-19}$ C
$\eta_o \cong 377$ ohms $= (\mu_o/\varepsilon_o)^{0.5}$
$(\nabla^2 - \mu\varepsilon\partial^2/\partial t^2)\bar{E} = 0$ [Wave Eqn.]

$E_y(z,t) = E_+(z-ct) + E_-(z+ct) = R_e\{\underline{E}_y(z)e^{j\omega t}\}$

$H_x(z,t) = \eta_o^{-1}[E_+(z-ct)-E_-(z+ct)]$ [or(ωt-kz) or (t-z/c)]

$\int_A (\bar{E} \times \bar{H}) \bullet d\bar{a} + (d/dt)\int_V \left(\varepsilon|\bar{E}|^2/2 + \mu|\bar{H}|^2/2\right) dv$
$= -\int_V \bar{E} \bullet \bar{J}\ dv$ (Poynting Theorem)

Media and Boundaries

$\bar{D} = \varepsilon_o \bar{E} + \bar{P}$

$\nabla \bullet \bar{D} = \rho_f,\ \tau = \varepsilon/\sigma$

$\nabla \bullet \varepsilon_o \bar{E} = \rho_f + \rho_p$

$\nabla \bullet \bar{P} = -\rho_p,\ \bar{J} = \sigma \bar{E}$

$\bar{B} = \mu \bar{H} = \mu_o(\bar{H} + \bar{M})$

$\varepsilon(\omega) = \varepsilon(1 - \omega_p^2/\omega^2),\ \omega_p = (Ne^2/m\varepsilon)^{0.5}$ (**plasma**)

$\varepsilon_{eff} = \varepsilon(1 - j\sigma/\omega\varepsilon)$

Skin depth $\delta = (2/\omega\mu\sigma)^{0.5}[m]$

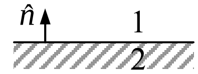
$\bar{E}_{1//} - \bar{E}_{2//} = 0$

$\bar{H}_{1//} - \bar{H}_{2//} = \bar{J}_s \times \hat{n}$

$B_{1\perp} - B_{2\perp} = 0$

$(D_{1\perp} - D_{2\perp}) = \rho_s$

($\bar{E}_{2//}$, $\bar{H}_{2//}$, $B_{2\perp}$, $D_{2\perp}$) $\rightarrow$ 0 = if $\sigma = \infty$



Electromagnetic Quasistatics

$\bar{E} = -\nabla\Phi(r), \Phi(r) = \int_{V'}(\rho(\bar{r})/4\pi\varepsilon|\bar{r}' - \bar{r}|)dv'$

$\nabla^2\Phi = \frac{-\rho_f}{\varepsilon}$

C = Q/V = Aε/d [F]
L = Λ/I
i(t) = C dv(t)/dt
v(t) = L di(t)/dt = dΛ/dt
$w_e = Cv^2(t)/2$; $w_m = Li^2(t)/2$
$L_{solenoid} = N^2\mu A/W$
$\tau = RC$, $\tau = L/R$
$\Lambda = \int_A \bar{B} \bullet d\bar{a}$ (per turn)
KCL : $\sum_i I_i(t) = 0$ at node
KVL : $\sum_i V_i(t) = 0$ around loop
$Q = \omega_0 w_T / P_{diss} = \omega_0/\Delta\omega$
$\omega_0 = (LC)^{-0.5}$
$\langle V^2(t) \rangle / R = kT$

Electromagnetic Waves

$(\nabla^2 - \mu\varepsilon\partial^2/\partial t^2)\bar{E} = 0$ [Wave Eqn.]

$(\nabla^2 + k^2)\hat{\bar{E}} = 0,\ \hat{\bar{E}} = \hat{\bar{E}}_o e^{-j\bar{k}\bullet\bar{r}}$

$k = \omega(\mu\varepsilon)^{0.5} = \omega/c = 2\pi/\lambda$

$k_x^2 + k_y^2 + k_z^2 = k_o^2 = \omega^2\mu\varepsilon$

$v_p = \omega/k,\ v_g = (\partial k/\partial\omega)^{-1}$

$\theta_r = \theta_i$
$\sin\theta_t / \sin\theta_i = k_i / k_t = n_i / n_t$
$\theta_c = \sin^{-1}(n_t / n_i)$
$\theta_B = \tan^{-1}(\varepsilon_t / \varepsilon_i)^{0.5}$ for TM
$\theta > \theta_c \Rightarrow \hat{\bar{E}}_t = \hat{\bar{E}}_i T e^{+\alpha x - jk_z z}$
$\bar{k} = \bar{k}' - j\bar{k}''$
$\Gamma = T - 1$

$T_{TE} = 2/(1 + [\eta_i \cos\theta_t / \eta_t \cos\theta_i])$

$T_{TM} = 2/(1 + [\eta_t \cos\theta_t / \eta_i \cos\theta_i])$

Radiating Waves

$$\nabla^2 \overline{A} - \frac{1}{c^2}\frac{\partial^2 \overline{A}}{\partial t^2} = -\mu J_f$$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho_f}{\varepsilon}$$

$$\overline{A} = \int_{V'} \frac{\mu J_f \left(t - r_{QP}/c\right) dV'}{4\pi r_{QP}}$$

$$\Phi = \int_{V'} \frac{\rho_f \left(t - r_{QP}/c\right) dV'}{4\pi\varepsilon r_{QP}}$$

$$\overline{E} = -\nabla\Phi - \frac{\partial \overline{A}}{\partial t},\ \overline{B} = \nabla \times \overline{A}$$

$$\nabla^2 \hat{\Phi} + \omega^2 \mu\varepsilon \hat{\Phi} = -\hat{\rho}/\varepsilon,$$

$$\Phi(x,y,z,t) = \mathrm{Re}\left[\hat{\Phi}(x,y,z)e^{j\omega t}\right]$$

$$\nabla^2 \hat{\overline{A}} + \omega^2 \mu\varepsilon \hat{\overline{A}} = -\mu\hat{\overline{J}},$$

$$\overline{A}(x,y,z,t) = \mathrm{Re}\left[\hat{\overline{A}}(x,y,z)e^{j\omega t}\right]$$

$$\hat{\Phi}(r) = \int_{V'} \hat{\rho}(\overline{r})e^{-jk|\overline{r}'-\overline{r}|}/\left(4\pi\varepsilon|\overline{r}'-\overline{r}|\right)\mathrm{dV}'$$

$$\hat{\overline{A}}(\mathrm{r}) = \int_{V'}\left(\mu\hat{\overline{J}}(\overline{r})e^{-jk|\overline{r}'-\overline{r}|}/4\pi|\overline{r}'-\overline{r}|\right)\mathrm{dV}'$$

$$\hat{E}_{ff\theta} = \sqrt{\frac{\mu}{\varepsilon}}\hat{H}_{ff\phi} = \left(j\eta k\hat{I}d/4\pi r\right)e^{-jkr}\sin\theta$$

Forces, Motors, and Generators

$$\overline{J} = \sigma(\overline{E} + \overline{v}\times\overline{B})$$

$\overline{F} = \overline{I}\times\overline{B}\ [\mathrm{Nm}^{-1}]$ (force per unit length)

$\overline{E} = -\overline{v}\times\overline{B}$ inside perfectly conducting wire $(\sigma \to \infty)$

Max f/A = $B^2/2\mu$, $D^2/2\varepsilon$ [Nm^{-2}]

$$vi = \frac{dw_T}{dt} + f\frac{dz}{dt}$$

$$f = ma = d(mv)/dt$$

$P = fv = T\omega$ (Watts)

$$T = I\, d\omega/dt$$

$$I = \sum_i m_i r_i^2$$

$\overline{F}_E = \lambda\overline{E}\left[Nm^{-1}\right]$ Force per unit length on line charge λ

$$W_M(\lambda,x) = \frac{1}{2}\frac{\lambda^2}{L(x)};\ W_E(q,x) = \frac{1}{2}\frac{q^2}{C(x)}$$

$$f_M(\lambda,x) = -\left.\frac{\partial W_M}{\partial x}\right|_\lambda = -\frac{1}{2}\lambda^2\frac{d}{dx}\left(1/L(x)\right) = \frac{1}{2}I^2\frac{dL(x)}{dx}$$

$$f_E(q,x) = -\left.\frac{\partial W_E}{\partial x}\right|_q = -\frac{1}{2}q^2\frac{d}{dx}\left(1/C(x)\right) = \frac{1}{2}v^2\frac{dC(x)}{dx}$$

Wireless Communications and Radar

$$G(\theta,\phi) = P_r/(P_R/4\pi r^2)$$

$$P_R = \int_{4\pi} P_r(\theta,\phi,r)\, r^2 \sin\theta\, d\theta d\phi$$

$$P_{rec} = P_r(\theta,\phi)A_e(\theta,\phi)$$

$$A_e(\theta,\phi) = G(\theta,\phi)\lambda^2/4\pi$$

$G(\theta,\phi) = 1.5\sin^2\theta$ (Hertzian Dipole)

$$R_r = P_R\Big/\left\langle i^2(t)\right\rangle = \frac{2\pi\sqrt{\mu/\varepsilon}}{3}\left(\frac{d\ell_{eff}}{\lambda}\right)^2$$

$$E_{ff}(\theta \cong 0) = \left(je^{jkr}/\lambda r\right)\int_A E_t(x,y)e^{jk_x x + jk_y y}dxdy$$

$\hat{E}_Z = \sum_i a_i \overline{E}e^{-jkr_i}$ = (element factor)(array f)

$E_{bit} \geq \sim 4 \times 10^{-20}$ [J]

$Z_{12} = Z_{21}$ if reciprocity

At ω_o, $\langle w_e\rangle = \langle w_m\rangle$

$$\langle w_e\rangle = \int_V \left(\varepsilon|\hat{\overline{E}}|^2/4\right)dv$$

$$\langle w_m\rangle = \int_V \left(\mu|\hat{\overline{H}}|^2/4\right)dv$$

$$Q_n = \omega_n w_{Tn}/P_n = \omega_n/2\alpha_n$$

$$f_{mnp} = (c/2)\left([m/a]^2 + [n/b]^2 + [p/d]^2\right)^{0.5}$$

$$s_n = j\omega_n - \alpha_n$$

Acoustics

$$P = P_o + p,\ \overline{U} = 0 + \overline{u}$$

$$\nabla p = -\rho_o \partial\overline{u}/\partial t$$

$$\nabla \bullet \overline{u} = -\left(1/\rho_o c_s^2\right)\partial p/\partial t$$

$$\left(\nabla^2 - k^2\partial^2/\partial t^2\right)p = 0$$

$$k^2 = \omega^2/c_s^2 = \omega^2\rho_o/\gamma P_o$$

$c_s = v_p = v_g = \left(\gamma P_o/\rho_o\right)^{0.5}$ or $\left(K/\rho_o\right)^{0.5}$ or $\sqrt{RT}$

$\eta_s = p/u = \rho_o c_s = (\rho_o\gamma P_o)^{0.5}$ or $\rho_0\sqrt{RT}$ gases

$\eta_s = (\rho_o K)^{0.5}$ solids, liquids

$p, \overline{u}_\perp$ continuous at boundaries

$$\hat{p}(z) = \hat{p}_+e^{-jkz} + \hat{p}_-e^{+jkz},\ p(z,t) = \mathrm{Re}\left[\hat{p}(z)e^{j\omega t}\right]$$

$$\hat{u}_z = \eta_s^{-1}(\hat{p}_+e^{-jkz} - \hat{p}_-e^{+jkz}),\ u_z(z,t) = \mathrm{Re}\left[\hat{u}_z(z)e^{j\omega t}\right]$$

$$\int_A \overline{u}p \bullet d\overline{a} + (d/dt)\int_V\left(\rho_o|\overline{u}|^2/2 + p^2/2\gamma P_o\right)dV$$

Optical Communications

$E = hf$, photons or phonons

hf/c = momentum [kg ms^{-1}]

$$dn_2/dt = -\left[An_2 + B\left(n_2 - n_1\right)\right]$$

Transmission Lines

Time Domain

$$\partial v(z,t)/\partial z = -L\partial i(z,t)/\partial t$$

$$\partial i(z,t)/\partial z = -C\partial v(z,t)/\partial t$$

$$\partial^2 v/\partial z^2 = LC\, \partial^2 v/\partial t^2$$

$$v(z,t) = V_+(t - z/c) + V_-(t + z/c)$$

$$i(z,t) = Y_o[V_+(t - z/c) - V_-(t + z/c)]$$

$$c = (LC)^{-0.5} = (\mu\varepsilon)^{-0.5}$$

$$Z_o = Y_o^{-1} = (L/C)^{0.5}$$

$$\Gamma_L = V_-/V_+ = (R_L - Z_o)/(R_L + Z_o)$$

Frequency Domain

$$(d^2/dz^2 + \omega^2 LC)\hat{V}(z) = 0$$

$$\hat{V}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz}\ ,\ v(z,t) = \mathrm{Re}\left[\hat{V}(z)e^{j\omega t}\right]$$

$$\hat{I}(z) = Y_0[\hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz}],\ i(z,t) = \mathrm{Re}\left[\hat{I}(z)e^{j\omega t}\right]$$

$$k = 2\pi/\lambda = \omega/c = \omega(\mu\varepsilon)^{0.5}$$

$$Z(z) = \hat{V}(z)/\hat{I}(z) = Z_o Z_n(z)$$

$$Z_n(z) = [1 + \Gamma(z)]/[1 - \Gamma(z)] = R_n + jX_n$$

$$\Gamma(z) = \left(V_-/V_+\right)e^{2jkz} = \left[Z_n(z) - 1\right]/\left[Z_n(z) + 1\right]$$

$$Z(z) = Z_o\left(Z_L - jZ_o \tan kz\right)/\left(Z_o - jZ_L \tan kz\right)$$

$$\mathrm{VSWR} = \left|V_{max}\right|/\left|V_{min}\right|$$