Frequency Spectra

What is your circuit radiating?

More Exactly

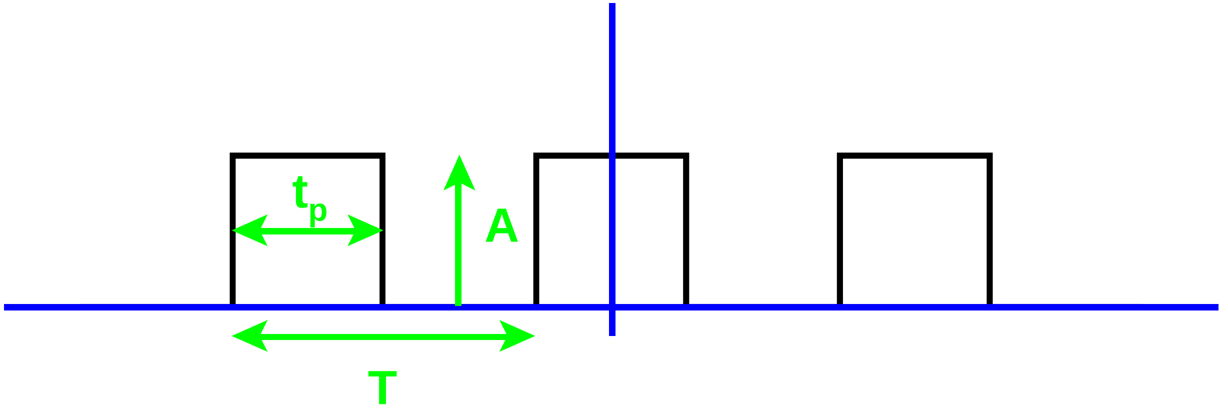
- What frequencies exist in your signal?
- What frequencies exist in your circuit?
- What can I consider to be the highest frequency of interest?

Waveforms

- Periodic
  - Sine or Cosine are most common
  - Pulse train is also very common
- Aperiodic, or single pulse
  - Also quite common

Pulse Train

- Rectangular Pulses
- Even Symmetry



Trigonometric Fourier Series

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{n=\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t$$

- Where the fundamental radian frequency is
  $\omega_o = 2\pi / T$

- Note presence of harmonics only

Harmonic Content Only?

- Look at a square wave constructed from Harmonics
- Add a non harmonic component
- Use Fourier for Pulse Train.mcd

Coefficients

- Harmonic amplitude coefficients
- D.C. Term

$$a_o = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_o t dt$$

- Zero for odd symmetry

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_o t dt$$

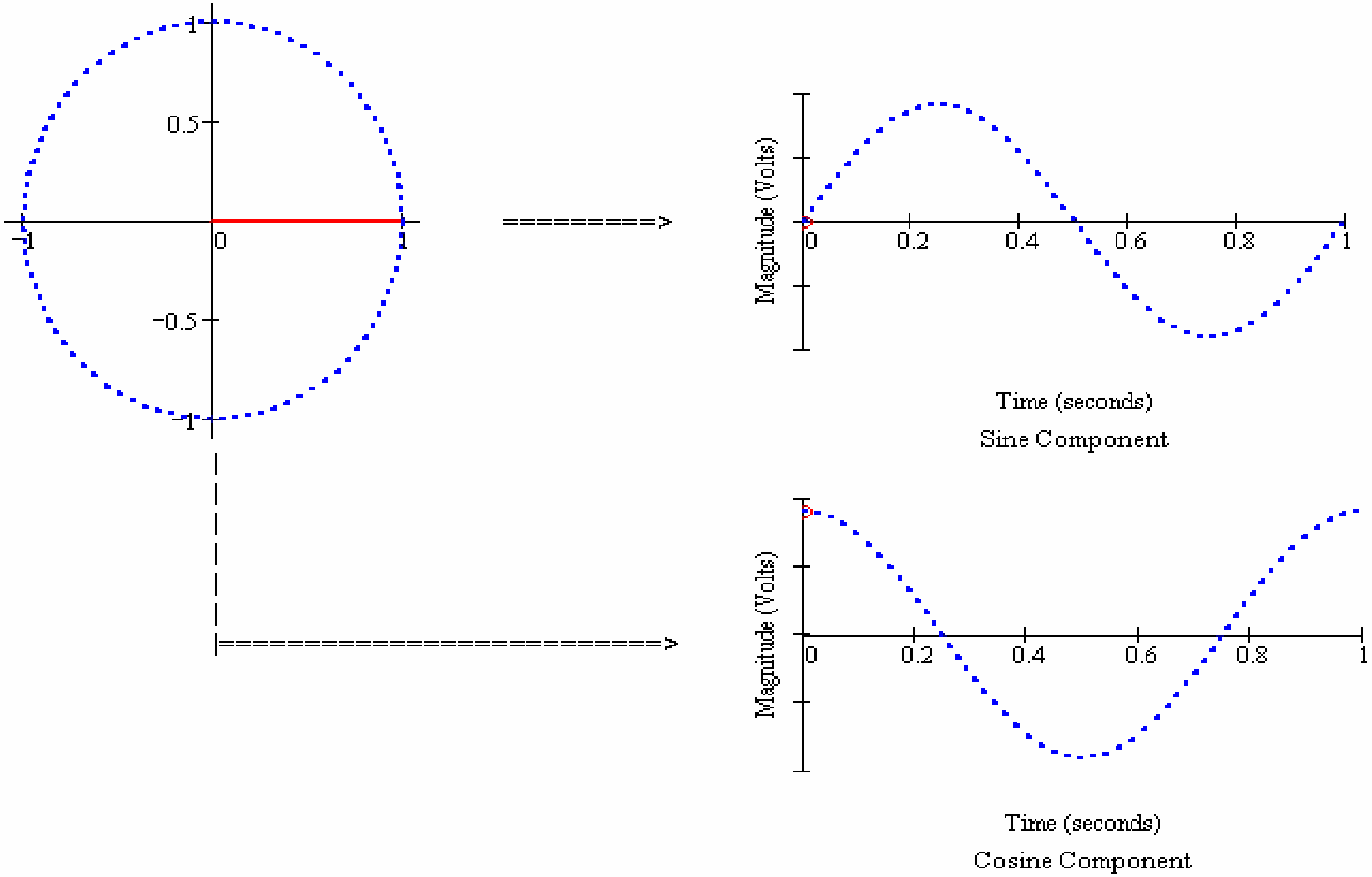
- Zero for even symmetry

Exponential Fourier Series

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_o t} = c_o + 2\sum_{n=1}^{n=\infty} |c_n| \left[\cos(n\omega_o t - \theta_n) + j\sin(n\omega_o t - \theta_n)\right]$$

- More compact
- Complex expression allows negative frequencies
  - Mathematical convenience

Exponential Form

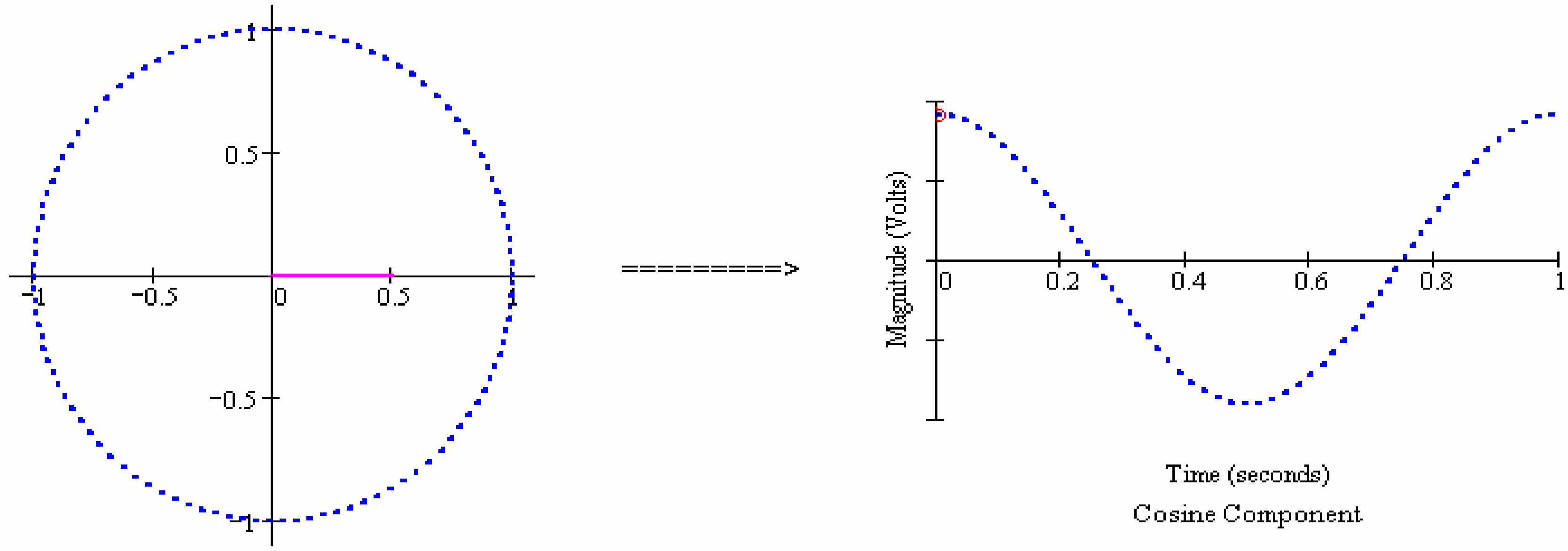


Coefficients

$$c_n = \frac{1}{2}(a_n - jb_n) \qquad c_n{}^* = \frac{1}{2}(a_n + jb_n)$$

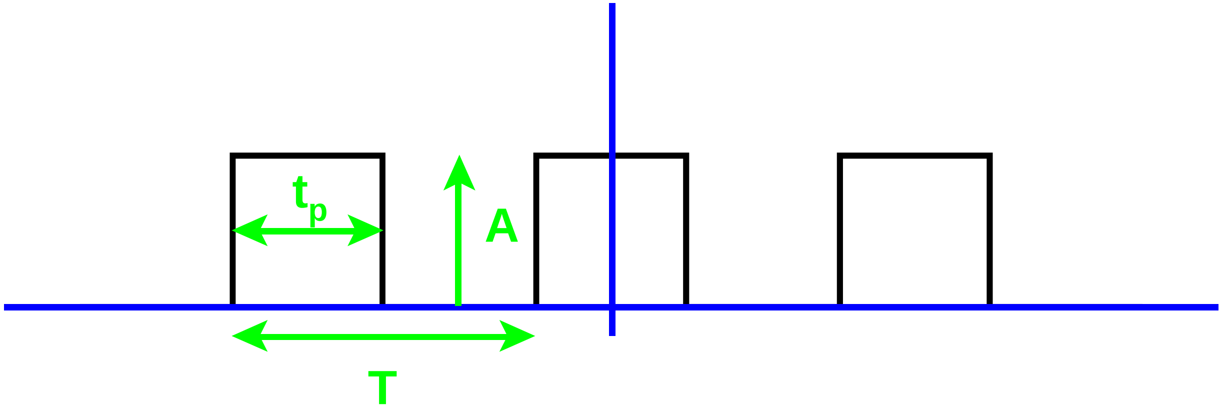
- Positive and Negative Frequencies
- Half amplitude of a_n and b_n

Negative Frequencies



Pulse Train

- Rectangular Pulses
- Even Symmetry



Pulse Train Harmonics

- Using Complex Form

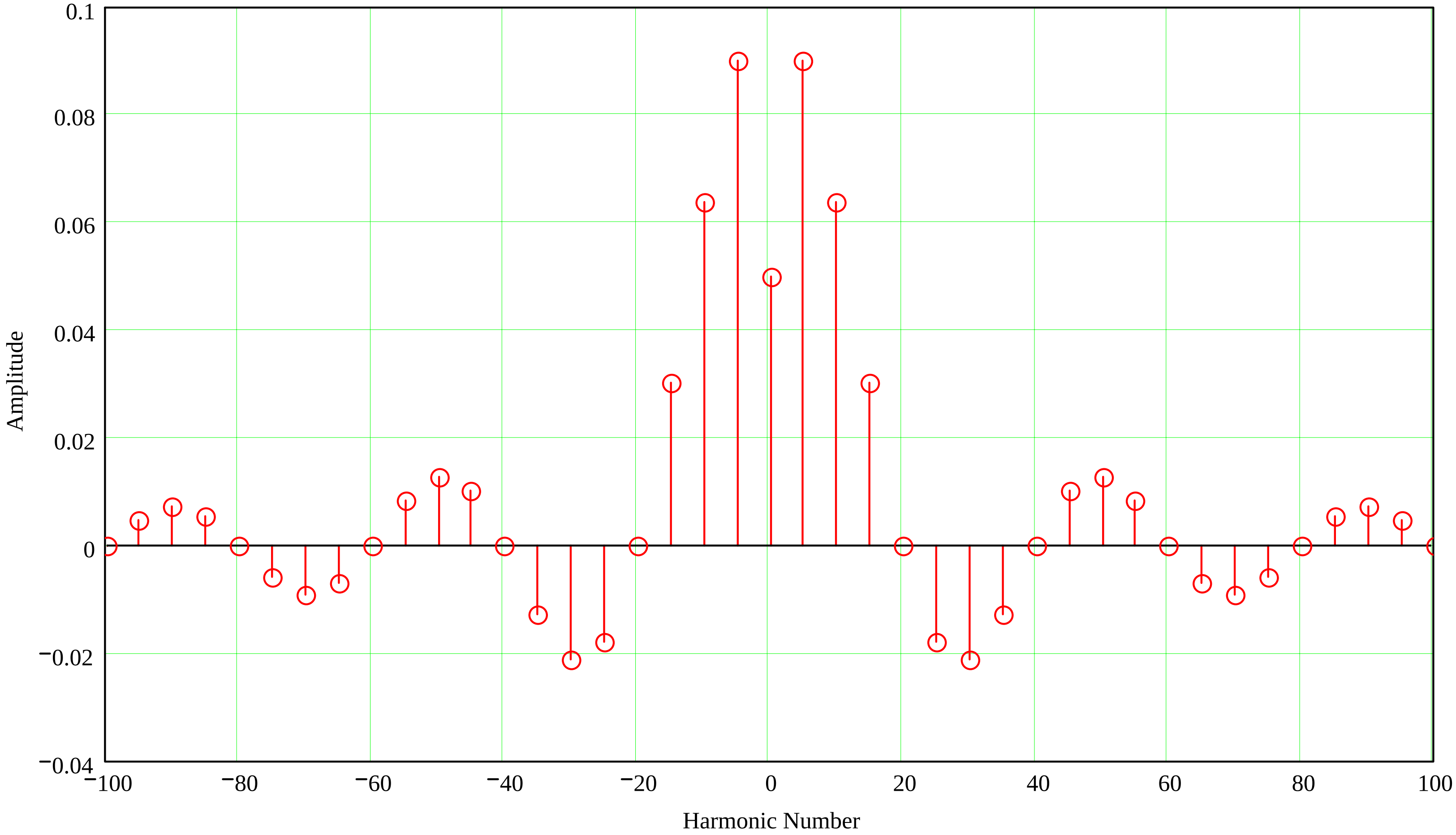
$$c_n = \frac{1}{T}\int_{-t_p/2}^{t_p/2} Ae^{-jn\omega_o t}dt = \frac{1}{T}\left[-\frac{A}{jn\omega_o}e^{-jn\omega_o t}\right]_{-t_p/2}^{t_p/2} = \frac{A}{T}\left[\frac{e^{+jn\omega_o {t_p}/{2}} - e^{-jn\omega_o {t_p}/{2}}}{jn\omega_o}\right]$$

- This gives

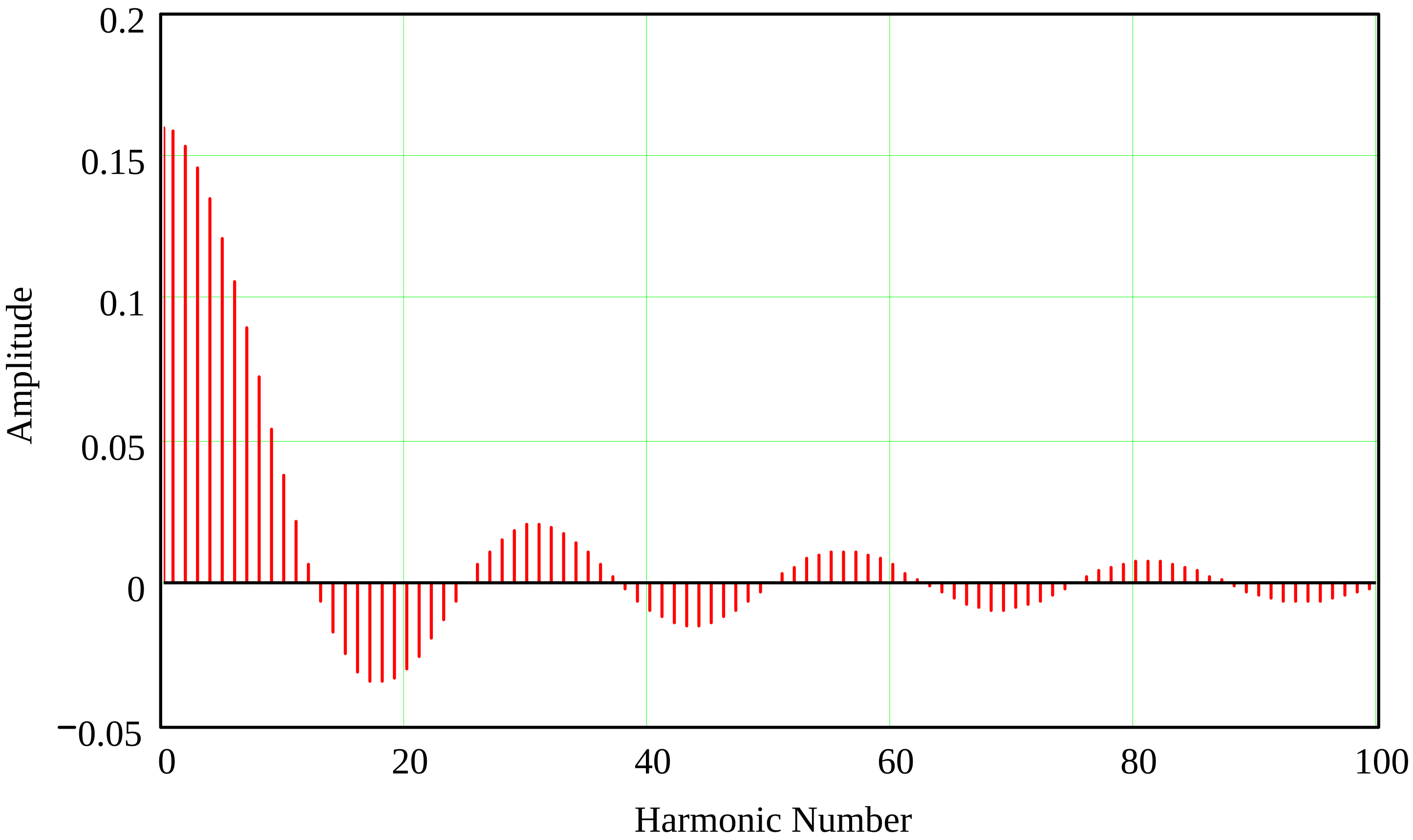
$$c_n = \frac{2A}{n\omega_o T}\sin(n\omega_o {t_p}/{2}) = \frac{At_p}{T}\frac{\sin(n\omega_o {t_p}/{2})}{n\omega_o {t_p}/{2}} = \frac{At_p}{T}\frac{\sin(n\pi f_o t_p)}{n\pi f_o t_p}$$

- Note Sinc(x) function giving double sided spectrum

Example Spectrum



Example Spectrum



Final Comments on Spectrum

- Even Symmetry
  - b terms zero
- $c_n = a_n/2$
- Spectrum becomes

$$f(t) = \frac{At_p}{T}\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin(n\pi f_0 t)}{n\pi f_0 t}\cos(n\omega_0 t)\right]$$

Final Comments on Spectrum

- DC Component

$$V_{DC} = \frac{At_p}{T}$$

- Duty Cycle

$$\delta = \frac{t_p}{T}$$

- DC

$$V_{DC} = A\delta$$

- Harmonic Amplitudes

$$a_n = 2A\delta \frac{\sin(n\pi\delta)}{n\pi\delta}$$

Points

- Harmonics separated by fundamental frequency
- The larger T, the closer the lines are in the spectrum
- Zero amplitudes occur when $f = 1/t_p$
- Negative amplitudes denote 180^o phase shift.

Single Pulse

- Aperiodic Waveform (Single Pulse)
- Fourier Series is not applicable
  - T is infinite
  - Separation of terms is 0Hz
- Continuous spectrum
- Fourier Transform used

Single Pulse

- No discrete frequencies
- Complimentary pair of transform equations.
- Pulse Definition

$$f(t) = \begin{cases} A & -\frac{t_p}{2} < t < \frac{t_p}{2} \\ 0 & elsewhere \end{cases}$$

Fourier Transform

- Transform Pair

$$c(f) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \int_{-\infty}^{\infty} c(f) e^{j\omega t} df$$

Single Pulse Spectrum

$$c(f) = A\int_{-\frac{t_p}{2}}^{\frac{t_p}{2}} e^{-j\omega t}\,dt$$

$$c(f) = \frac{A}{-j\omega}\Big[\cos(\omega t) - j\sin(\omega t)\Big]_{-\frac{t_p}{2}}^{\frac{t_p}{2}}$$

$$c(f) = \frac{2A}{\omega}\left[\sin(\omega\frac{t_p}{2})\right]$$

Single Pulse Spectrum

$$c(f) = At_p \left[\frac{\sin(\omega \frac{t_p}{2})}{\omega \frac{t_p}{2}} \right]$$

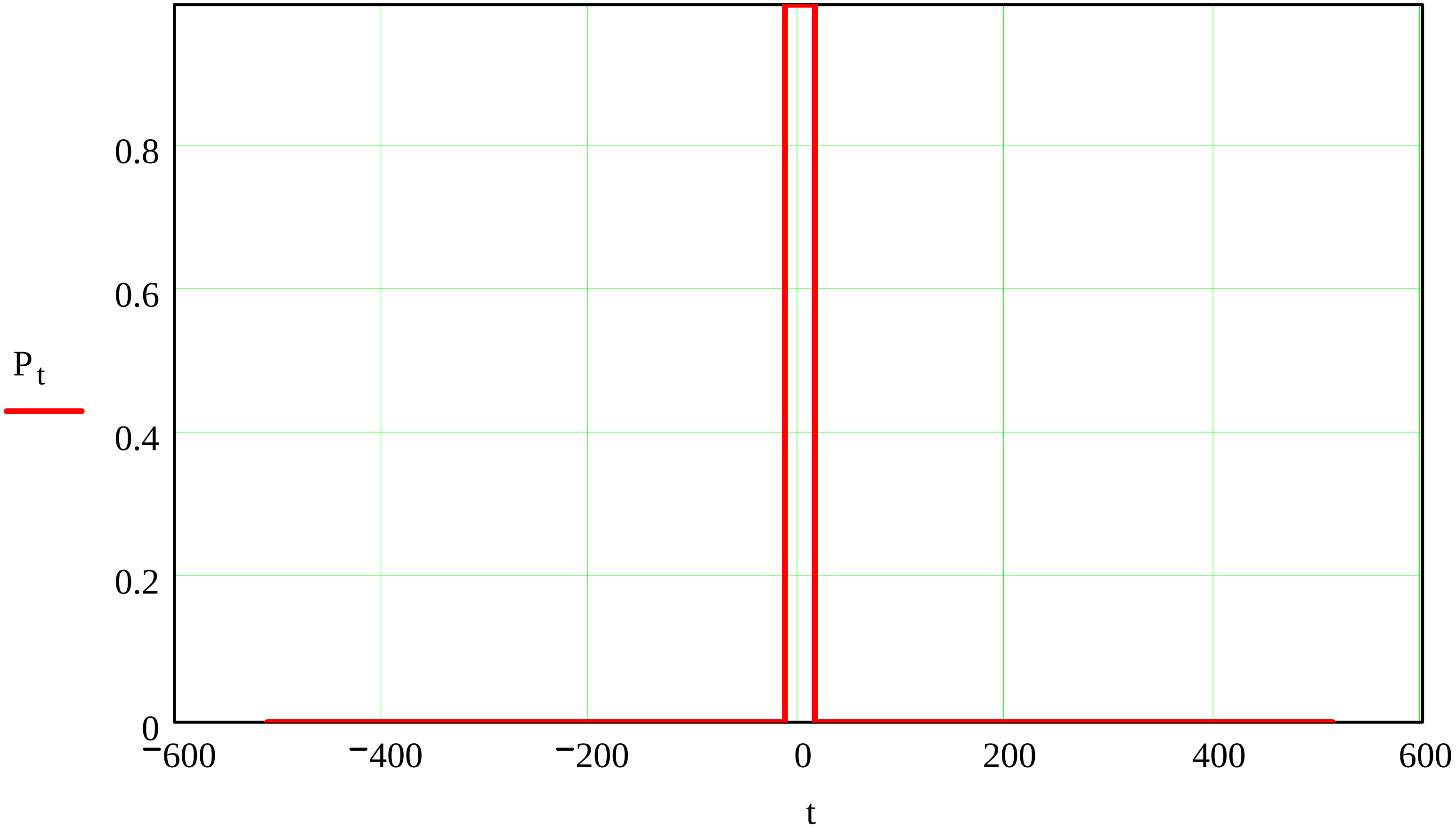
$$c(f) = At_p \left[\frac{\sin(\pi f t_p)}{\pi f t_p} \right]$$

$$c(f) = At_p \sin c(\pi f t_p)$$

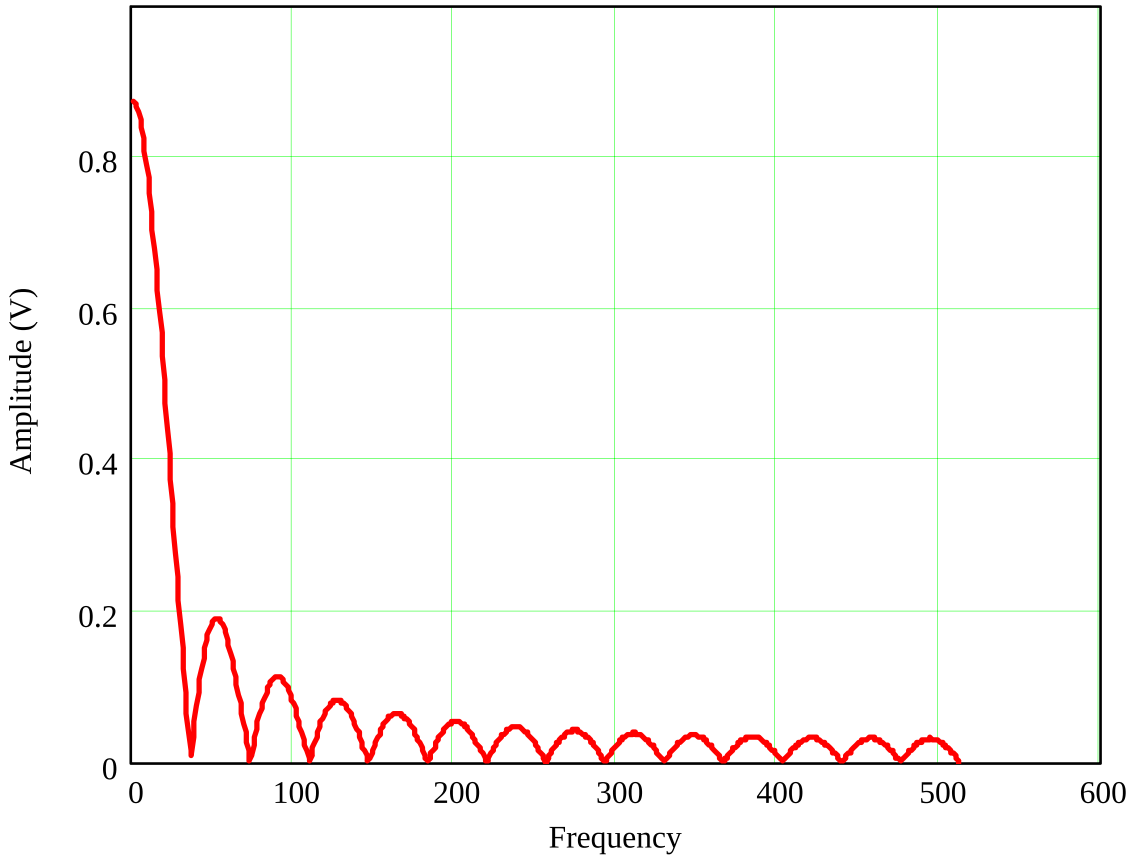
Single Pulse Spectrum

- Sinc tends to 1 as f tends to 0
- Amplitude becomes At_p at 0Hz
- Amplitude in V or V/Hz
  - Spectral Density
- Zero Crossings at $1/t_p$, $2/t_p$ etc.
- Occur at higher frequencies as t_p reduces

Single Pulse



Single Pulse Spectrum



Single Pulse Spectrum

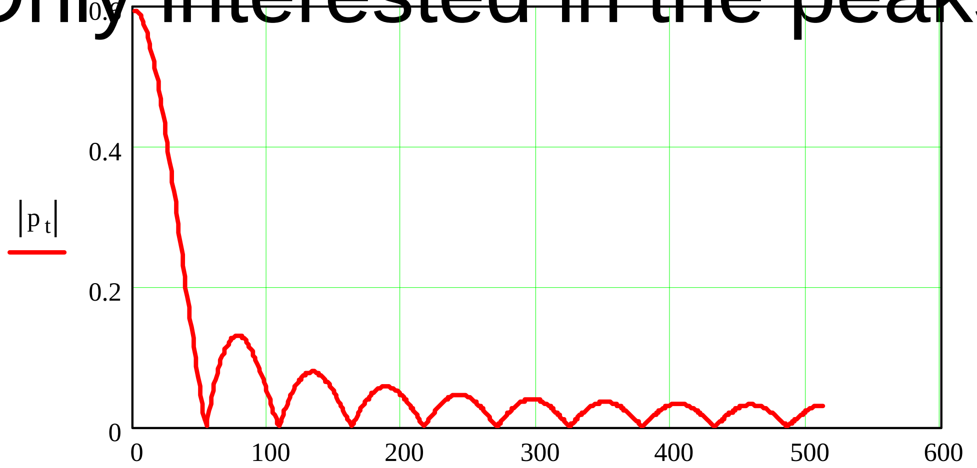
- Formulae work well for theoretical calculation
- Measurements are slightly different.
  - Sampled data
  - No neat equation for pulse (stream)
- Use Fast Fourier Transform (FFT)
  - Available in most maths packages
  - Available in many oscilloscopes

Some Practicalities

- What is the highest frequency component I need to worry about?
- Depends on spectrum of signal
- Depends on frequency dependence of coupling
- Look here at the maxima of spectrum
- Produce a fairly simple design graph

Pulse Train Approximate Harmonic Envelope

- Only interested in the peaks of the lobes



$$\sin(n\pi f_0 t_p) = \pm 1$$

Pulse Train Approximate Harmonic Envelope

- This gives us

$$n\pi f_0 t_p = \frac{\pi}{2} \quad \frac{3\pi}{2} \quad \frac{5\pi}{2} \quad etc$$

- at frequencies

$$f = nf_0 = \frac{1}{2t_p} \quad \frac{3}{2t_p} \quad \frac{5}{2t_p} \quad etc$$

Pulse Train Approximate Harmonic Envelope

- Approximate envelope of the peaks follows
  $$f = \frac{1}{\pi t_p}$$
- a_{nmax} is $\frac{2A}{n\pi}$
- This comes from $$\left|a_{n_{\max}}\right| = 2\frac{t_p}{T}A\frac{1}{\pi n f_o t_p}$$

Pulse Train Approximate Harmonic Envelope

- Largest Amplitude occurs at the lowest frequency

$$\frac{1}{\pi f t_p} = 1 \quad or \quad f = \frac{1}{\pi t_p}$$

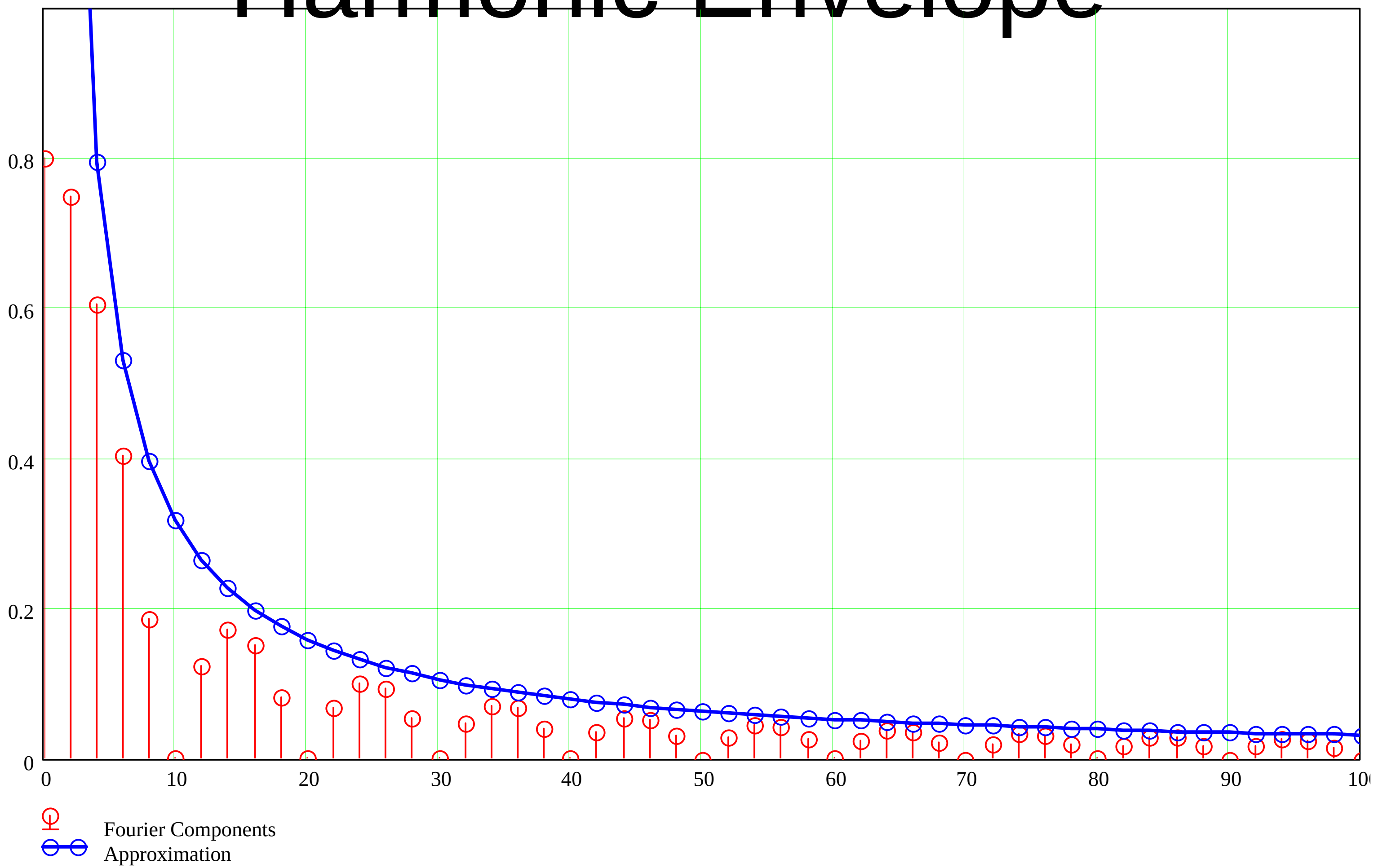
- The maximum amplitude is

$$\left| a_{n_{\max}} \right| = 2A\delta$$

Pulse Train Approximate Harmonic Envelope

- At higher frequencies $$\left|a_{n_{\max}}\right| = \frac{2A\delta}{\pi f\, t_p}$$
- This give an inverse frequency relationship

Pulse Train Approximate Harmonic Envelope



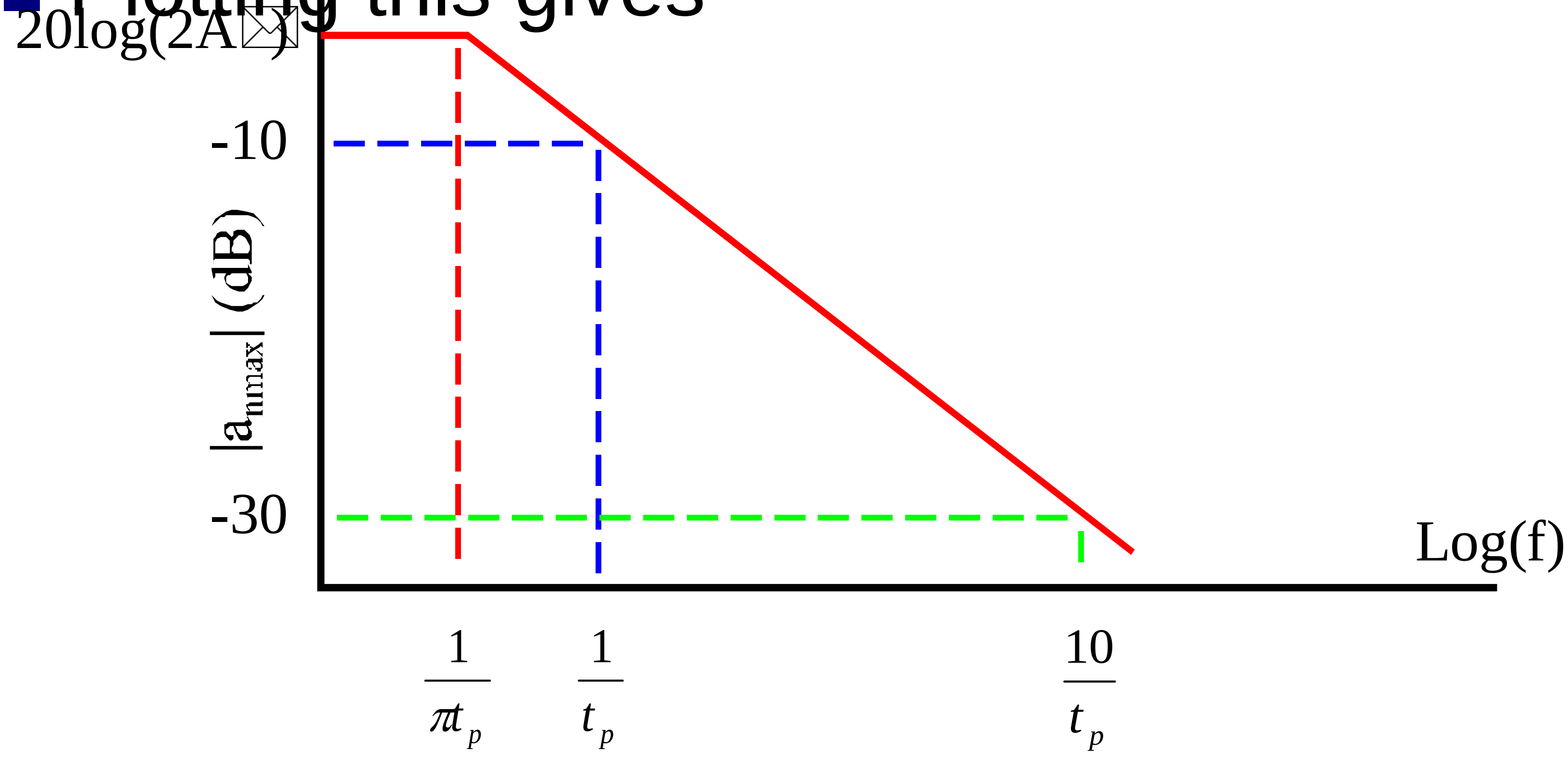
Pulse Train Approximate Harmonic Envelope

- Logarithmic form is more easily understood
- We get 0dB up to $f = \dfrac{1}{\pi t_p}$
- Equation becomes

$$\left|a_{n_{\max}}\right| = 20\log(2A\delta) - 20\log(\pi f t_p)$$

Pulse Train Approximate Harmonic Envelope

- Plotting this gives

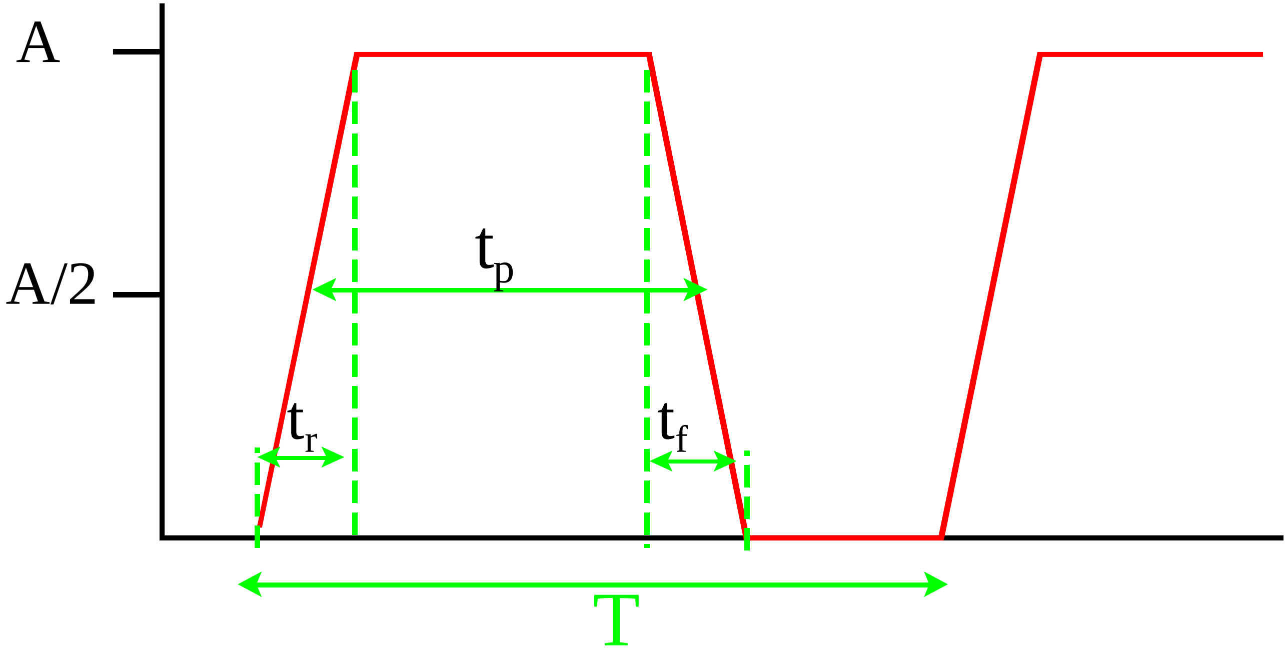


Applications

- Easier estimate of maximum levels of frequency components of a digital signal
- Allows estimation of maximum harmonic that needs to be considered
  - Must take into account frequency dependence of coupling in some cases.

Pulses with Finite Rise and Fall Times

- We have a trapezoidal pulse



Pulses with Finite Rise and Fall Times

- Pulse width is average pulse width
- Rise and Fall times are equal
- Spectrum contains a second Sinc term

$$a_n = 2A\delta \left[\frac{\sin(n\pi f_0 t_p)}{n\pi f_0 t_p} \right] \left[\frac{\sin(n\pi f_0 t_r)}{n\pi f_0 t_r} \right]$$

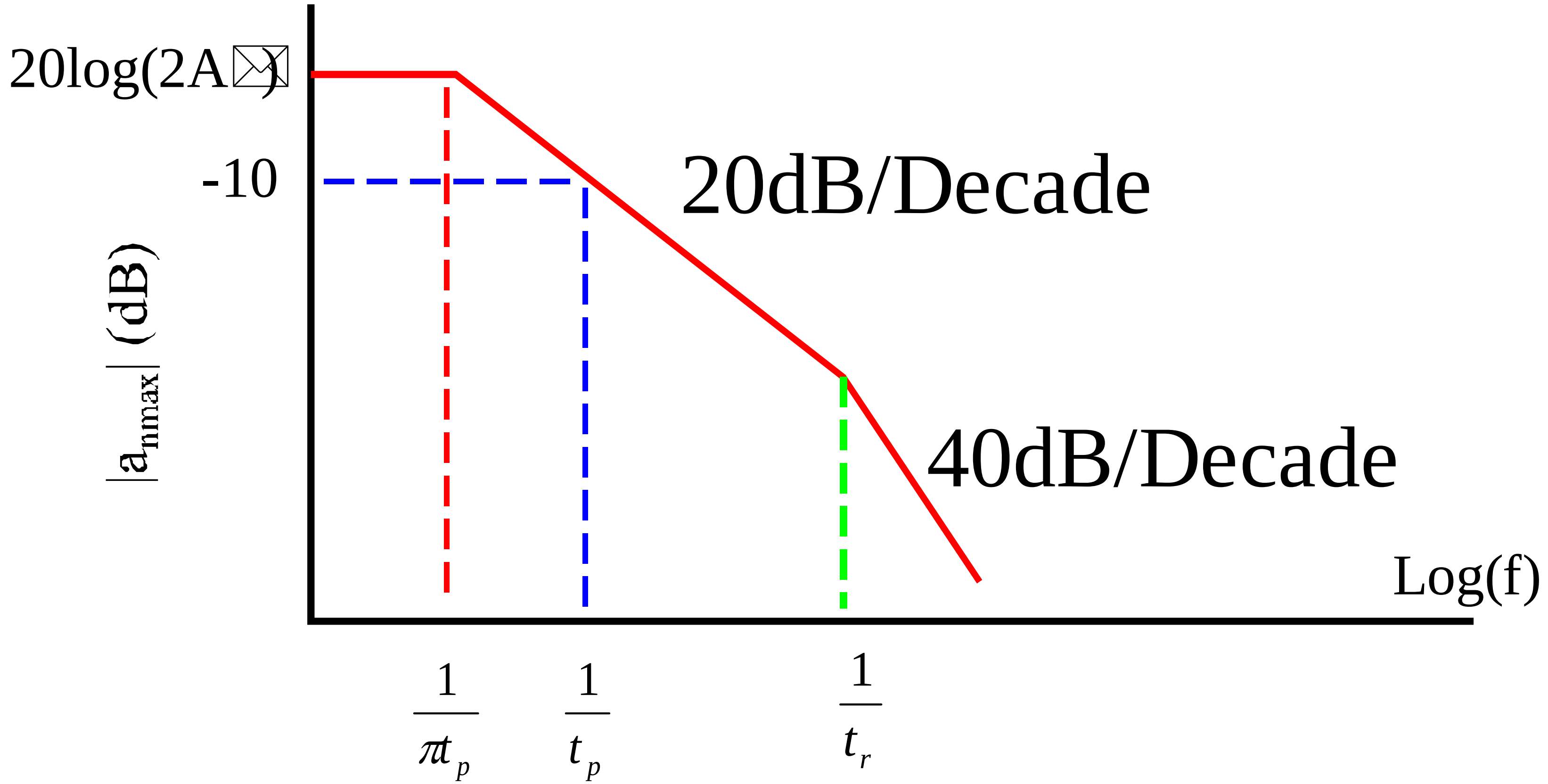
Pulses with Finite Rise and Fall Times

- Maxima given by

$$\left|a_{n_{\max}}\right| = 20\log(2A\delta) - 20\log(\pi f t_p) - 20\log(\pi f t_r)$$

- where $f=nf_0$ again
- There are two break frequencies

Pulses with Finite Rise and Fall Times



Conclusions

- Virtually all signals are complex
  - They contain harmonics and other frequencies
- Fourier Analysis/ gives spectrum for mathematically expressible signals
- FFT processes measured data
- Simple graphs allow estimation of frequencies to consider